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Symmetries in Quantum Mechanics.

→ In Classical mechanics:

Given a Lagrangian $L(q,\dot{q},t)$

If under $q \to q + \delta q$,

$$\frac{\delta L}{\delta q} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\Rightarrow \boxed{\frac{dp}{dt} = 0} \quad , \text{ where } \underbrace{p \equiv \frac{\partial L}{\partial \dot{q}}}_{\text{Conjugate momenta}}$$

$$\Rightarrow p = p_0 \leftarrow \text{Conserved quantity}$$

⇒ For every symmetry of the Lagrangian, ∃ a conserved quantity.

$\boxed{\text{Noether's theorem}}$

Q How to think about symmetries in QM?

→ Transformations ??

→ What is invariant under such transformations ??

→ What is the conserved quantity ??

→ Note that, In QM states are vectors in Hilbert space & $\exists$ unitary transformations relating any two vectors in the Hilbert space.

→ Any transformation that describes a symmetry in QM should be given by some unitary operator U, that leaves the Schroedinger eq. unchanged !!

~~→ As an example let's consider first a classical system with rotational symmetry transformation.~~

→ The "Algorithm" we hope to follow:

(1) Identify a unitary operator that ~~perf~~ effects a transformation in the Quantum sys.

(2) If the ~~H~~ Hamiltonian or Schroedinger eq. remains unchanged under this transformation, then there exists a symmetry.

(3) Corresponding to this symmetry, identify the "generator" & degeneracies in the system.

→ Transformation Operators in QM

Let's consider the simplest example of translation.

Suppose $\exists \; \hat{T} : \hat{T}|\psi(\vec{r})\rangle = |\psi(\vec{r}+\vec{x})\rangle$

To construct this "Translation" operator, let's first consider infinitesimal translation:

$$\hat{T}(\Delta x)\psi(x) = \psi(x+\Delta x)$$

$$\Rightarrow \quad \hat{T}(\Delta x)\psi(x) - \psi(x) = \psi(x+\Delta x) - \psi(x)$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\hat{T}(\Delta x) - 1\right]\psi(x) = \lim_{\Delta x \to 0} \frac{\psi(x+\Delta x) - \psi(x)}{\Delta x}$$

$$\Rightarrow \quad \frac{1}{\Delta x}\left[\hat{T}(\Delta x) - 1\right]\psi(x) = \frac{\partial}{\partial x}\psi(x)$$

$$\Rightarrow \quad \cancel{\hat{T}(\Delta x)\psi(x) = \Delta x\left[1 + \frac{\partial}{\partial x}\right]\psi(x)}$$

$$\Rightarrow \quad \boxed{\hat{T}(\Delta x)\psi(x) = \left[1 + \Delta x \cdot \frac{\partial}{\partial x}\right]\psi(x)}$$

Illy in 3D:

$$\hat{T}(\Delta\vec{x})\psi(\vec{x}) = \left[1 + \Delta\vec{x}\cdot\vec{\nabla}\right]\psi(x)$$

$$= \left[1 + \frac{i}{\hbar}\Delta\vec{x}\cdot\underbrace{(-i\hbar\vec{\nabla})}_{\hat{\vec{p}}}\right]\psi(x)$$

$\therefore$ For a general infinitesimal ~~transf~~ transformation $\vec{\varepsilon}$:

$$\boxed{\hat{T}(\vec{\varepsilon}) = 1 + \frac{i}{\hbar}\hat{\varepsilon}\cdot\underbrace{\hat{\vec{p}}}_{}}$$

called the "generator" of translation.

From infinitesimal to finite translations:

→ First, check the unitarity of translation operator: $\boxed{H/W}$

→ Consider N applications of $\hat{T}$ operator, performing infinitesimal translations $\vec{\epsilon}$ N times.

i.e: $\hat{T}(\vec{\epsilon}) \ldots \hat{T}(\vec{\epsilon}) |\psi\rangle$

where $N\vec{\epsilon} = \vec{x}$

$$\Rightarrow \lim_{N\to\infty} [\hat{T}(\epsilon)]^N |\psi\rangle = \left[\hat{T}\left(\frac{\vec{x}}{N}\right)\right]^N |\psi\rangle$$

$$= \lim_{N\to\infty} \left[1 + \frac{i}{\hbar} \frac{\vec{x}\cdot\vec{p}}{N}\right]^N |\psi\rangle$$

$$\Rightarrow \boxed{\hat{T}(\vec{x}) = \lim_{N\to\infty}\left[1 + \frac{i}{\hbar}\frac{\vec{x}\cdot\hat{\vec{p}}}{N}\right]^N = \exp\left(\frac{i}{\hbar}\vec{x}\cdot\hat{\vec{p}}\right)}$$

→ NOTE: Due to sign of exp(...), it is customary to assign $\hat{T}(\vec{x}) \equiv \hat{D}^\dagger(\vec{x})$ ← Spatial translation operator.

$$\Rightarrow \boxed{\hat{D}(\vec{x}) = \exp\left(-\frac{i}{\hbar}\vec{x}\cdot\hat{\vec{p}}\right)} = \lim_{N\to\infty}\left(1 - \frac{i}{\hbar}\frac{\vec{x}\cdot\hat{\vec{p}}}{N}\right)^N$$

$\boxed{H/W}$ Check that under $\vec{x}\to 0$:

$$\boxed{\hat{D}(\vec{x}) \approx 1 - \frac{i}{\hbar}\vec{x}\cdot\vec{p}}$$

→ The takeaway:

→ Momentum is the generator of translation.

Ily → Angular momentum is the generator of rotation!!

Let's construct the angular momentum operator!

(Note: $\hat{L} \neq \hat{r} \times \hat{p}$ in QM)

Let, J_k is the generator of infinitesimal rotation around the k^{th} axis.

i.e $\varepsilon \to d\phi$; $p_k \to J_k$

Hence, the rotation operator:

$$\Rightarrow \quad \mathcal{D}_k \equiv 1 - \frac{i}{\hbar} \hat{J}_k \, d\phi$$

Or, in terms of unit vector $\hat{n}$ along k-axis:

$$\boxed{\mathcal{D}(\hat{n}, d\phi) = 1 - \frac{i}{\hbar}(\hat{n} \cdot \hat{J})\, d\phi}$$

where $\hat{J} \to$ total angular momentum op.

∴ Finite rotation $\phi = \lim_{N \to \infty}(N\, d\phi)$ implies, along z-axis:

$$\mathcal{D}_z(\phi) = \left[1 - \frac{i}{\hbar} \frac{J_z \cdot \phi}{N}\right]^N = \exp\left(-\frac{i}{\hbar}\phi J_z\right)$$

$$\therefore \quad \boxed{\mathcal{D}_z(\phi) \equiv \exp\left(-\frac{i}{\hbar}\phi J_z\right)}$$

$\rightarrow$ Symmetry:

Given a transformation operator, say $\hat{S}$, ~~and~~ corresponding to generator G, given by

$$S = 1 - \frac{i\epsilon}{\hbar} \underbrace{G}_{\rightarrow \text{Hermitian.}}$$

If the Hamiltonian is invariant under the transformation S, then

under $|\psi\rangle \longrightarrow S|\psi\rangle$

$$\boxed{H|\psi\rangle = E|\psi\rangle \text{ is invariant.}}$$

$$\Rightarrow \quad H|S\psi\rangle = HS|\psi\rangle = SH|\psi\rangle = ES|\psi\rangle$$

$$\Rightarrow \quad \cancel{HS - SH = 0}$$

$$\Rightarrow \quad \langle S\psi|H|S\psi\rangle = E$$

$$\& \quad \langle\psi|S^{\dagger}HS|\psi\rangle = \langle\psi|H|\psi\rangle$$

$$\& \quad \boxed{S^{\dagger}HS = H}$$

$$\Rightarrow \left(1 + \frac{i\epsilon}{\hbar}G\right) H \left(1 - \frac{i\epsilon}{\hbar}G\right) = H$$

$$\Rightarrow \cancel{H} + \left[\frac{i\epsilon}{\hbar}GH - \frac{i\epsilon}{\hbar}HG\right] + O(\epsilon^2) = \cancel{H}$$

$$\Rightarrow \frac{i\epsilon}{\hbar}(GH - HG) = 0$$

$$\Rightarrow \boxed{[G, H] = 0}$$

Using the Heisenberg eq. of motion:

$$\dot{G} = \frac{-i}{\hbar}[G, H]$$

$$\Rightarrow \boxed{\dot{G} = 0}$$

$\therefore$

G is the conserved quantity !!

As expected from our algorithm !!

→ Compared to Classical case in QM we encounter an additional effect of ~~conserve~~ symmetry:

"<u>Degeneracy</u>".

$\because [H, S] = 0$

$$\Rightarrow \quad H(S|n\rangle) = S(H|n\rangle) = E_n(S|n\rangle)$$

$\Rightarrow$ For distinct $|n\rangle$ & $S|n\rangle$, the eigenvalues are identical $\Leftarrow$ <u>degeneracy</u>!