Discrete Symmetries.

→ Symmetries need not be $\boxed{\text{continuous}}$!

corresponds to operations obtained by successive applications of infinitesimal symmetry operations.

→ In contrast, discrete symmetry operators can't have infinitesimal transformation.
<u>eg:</u> Parity, time-reversal, lattice translation.

Parity

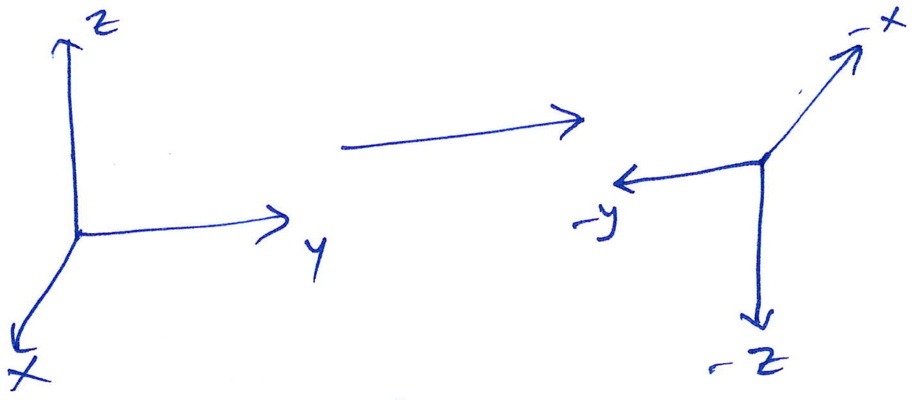
Consider a unitary operator "$\hat{P}$" acting on some eigenstate $|\alpha\rangle$:

$$|\alpha\rangle \longrightarrow \hat{P}|\alpha\rangle$$

→ We define the parity operator $\hat{P}$, s.t.

$$\hat{P}\,|\alpha(\vec{x})\rangle = |\alpha(-\vec{x})\rangle$$

i.e A LH coordinate ⟶ RH coordinate.



~~$\therefore \langle \psi(\vec{x})| \hat{P}^\dagger \vec{x} \hat{P} |\psi(\vec{x})\rangle = \langle \psi(-\vec{x})| \vec{x} |\psi(-\vec{x})\rangle$~~

Q. The expectation value of $\hat{\vec{x}}$ under parity transformation?

$$\langle \hat{\vec{x}} \rangle \xrightarrow[\text{parity}]{\text{under}} -\langle \hat{\vec{x}} \rangle$$

$$\Rightarrow \langle \alpha | \hat{P}^\dagger \hat{\vec{x}} \hat{P} | \alpha \rangle = -\langle \alpha | \hat{\vec{x}} | \alpha \rangle$$

$$\Rightarrow \hat{P}^\dagger \hat{\vec{x}} \hat{P} = -\hat{\vec{x}}$$

$$\Rightarrow \underbrace{\hat{P}\hat{P}^\dagger}_{\mathbb{1}} \hat{\vec{x}} \hat{P} = -\hat{P}\hat{\vec{x}}$$

$$\Rightarrow \hat{\vec{x}} \hat{P} = -\hat{P}\hat{\vec{x}}$$

$$\Rightarrow \boxed{\{\hat{\vec{x}}, \hat{P}\} = 0}$$

Also, Consider a state $|\vec{x}'\rangle$ (labeled by pos.)

Let, $\hat{P}|\vec{x}'\rangle$ is an eigen vector.

$$\Rightarrow \hat{\vec{x}} \hat{P} |\vec{x}'\rangle = -\hat{P}\hat{\vec{x}}|\vec{x}'\rangle$$

$$= -\hat{P}\vec{x}'|\vec{x}'\rangle \quad \longrightarrow \text{c-number},$$

$$= (-\vec{x}')\hat{P}|\vec{x}'\rangle$$

$\because \hat{\vec{x}}|\vec{x}'\rangle = \vec{x}'|\vec{x}'\rangle$, But, $\hat{\vec{x}}\hat{P}|\vec{x}'\rangle = (-\vec{x}')\hat{P}|\vec{x}'\rangle$

$$\Rightarrow \boxed{\hat{P}|\vec{x}'\rangle = |-\vec{x}'\rangle}$$

→ <u>Wave function under parity:</u>

Let, $\psi(\vec{x}') = \langle \vec{x}' | \alpha \rangle$

Then, $\langle \vec{x}' | \hat{P} | \alpha \rangle = \langle -\vec{x}' | \alpha \rangle = \psi(-\vec{x}')$

~~$\Rightarrow \hat{P}\,\psi(\vec{x}) =$~~

But, if $|\alpha\rangle$ is the eigenstate of $\hat{P}$,

$\Rightarrow \quad \hat{P} |\alpha\rangle = \pm |\alpha\rangle$

$\Rightarrow \quad \langle \vec{x}' | \hat{P} | \alpha \rangle = \pm \langle \vec{x}' | \alpha \rangle = \psi(-\vec{x}')$

$$\Rightarrow \quad \boxed{\psi(-\vec{x}') = \pm \psi(\vec{x}')}$$

→ If $\psi(-\vec{x}') = +\psi(\vec{x}')$ ← Even parity.

$\psi(-\vec{x}') = -\psi(\vec{x}')$ ← Odd parity.

→ Hermiticity of $\hat{P}$:

$$\because \quad \hat{P}|\vec{x}\rangle = |-\vec{x}\rangle$$

$$\Rightarrow \quad \hat{P}^2|\vec{x}\rangle = \hat{P}|-\vec{x}\rangle = |-(-\vec{x})\rangle$$

$$\Rightarrow \quad \boxed{\hat{P}^2 = \mathbb{1}} = \hat{P}^\dagger \hat{P}$$

$$\Rightarrow \quad \boxed{\hat{P}^\dagger = \hat{P}^{-1} = \hat{P}} \leftarrow \boxed{\text{Hermitian \& unitary.}}$$

→ Summary of properties of the Parity operator:

(1) $\hat{P}|\alpha\rangle = \lambda|\alpha\rangle$, $\lambda = \pm 1$ (eigenvalue.)

(2) $\hat{P}^2 = \mathbb{1}$

(3) $\{\hat{\vec{x}}, \hat{P}\} = 0$.

→ Example: Momentum operator under parity:

Recall — momentum is the generator of translation

→ $\vec{p} = \frac{d\vec{x}}{dt} \Rightarrow$ under $\vec{x} \longrightarrow -\vec{x}$:

$\vec{p} \longrightarrow -\vec{p}$ ller to $\vec{x}$!

$$\Rightarrow \boxed{\{\hat{P}, \hat{\vec{p}}\} = 0}$$

Alternative:

$$\hat{P}\,\hat{T}(d\vec{x}') = \hat{T}(-d\vec{x}')\,\hat{P}$$