"2nd Quantization" + STR

Recall: In Dirac's "2nd Quant." approach,

$|n\rangle \longrightarrow$ state with n photons.

$a^\dagger \longrightarrow$ "creates" a photon.

$a \longrightarrow$ "destroys" a photon.

However, the algebra is ‖er to SHO!

~~First~~

→ Now, our goal is to find a first-principles approach to finding the 2nd quantization algebra . .

→ Let's start with a classical theory of SHO:

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\Rightarrow \quad x(t) \sim e^{\pm i\omega t}, \quad \omega \equiv \sqrt{\frac{k}{m}}$$

Constructing the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

"First" Quantization

$$\boxed{\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2}$$

→ Then using the canonical commutation relation,

$$[\hat{x}, \hat{p}] = i \qquad [\hbar = 1, c = 1]$$

$$\Downarrow$$

$$[a, a^{\dagger}] = 1$$

where $a = \sqrt{\frac{m\omega}{2}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$; $a^{\dagger} = \sqrt{\frac{m\omega}{2}}\left(x - \frac{i\hat{p}}{m\omega}\right)$

$\boxed{Q}$ Is this consistent with STR ??

→ Check: $\hat{H}$ transformation under ~~LT~~ Lorentz trans ?

→ Check: The eq. of motion from $\hat{H}$:

$\Rightarrow \hat{H}\Psi = E\Psi$ ← How does this transform under Lorentz transformation?

$$\Rightarrow \left(-\nabla^2 + \frac{m\omega^2}{2}x^2\right)\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$\boxed{H/W}$ Find out how this eq. behaves when $x \to \gamma(x - vt)$; $t \to \gamma(t - vx)$

Also, suppose $\psi(x,t) = e^{\pm i \vec{p}\cdot\vec{x}} e^{\pm i E t}$

$$\Rightarrow \quad \hat{H}\psi = i\frac{\partial \psi}{\partial t}$$

$$\Rightarrow \left(-\nabla^2 + \frac{m\omega^2}{2}x^2\right)\psi = i\frac{\partial \psi}{\partial t}$$

$$\Rightarrow \boxed{p^2 + \frac{m\omega^2}{2}x^2 = E}$$

$\leftarrow$ Non-relativistic energy-momentum relation!!

$\rightarrow$ Not consistent with STR!!

[Q] How to construct a relativistic version of the Harmonic oscillator??

$\rightarrow$ The simplest Lorentz-invariant eq. of motion would be: ~~(modifying the Sch. TDSE)~~: [Taking hint from the behaviour of TDSE under L.T.]:

$$\boxed{\nabla^2\psi = \frac{\partial^2}{\partial t^2}\psi}$$

$\leftarrow$ where $V(x) = 0$ for convenience.

[H/W] Check for a 1+1 D case; that

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial t'^2}$$

whereas $\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ t \end{pmatrix}$

Therefore, in the tensor notation,

$$-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_i^2} \equiv \partial_\mu \partial^\mu \equiv \Box$$

$\partial_\mu \partial^\mu \downarrow \eta_{\mu\nu} \partial^\mu \partial^\nu$ $\qquad$ $\Box \uparrow$ de'Alembertian

where $\qquad \partial_\mu \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

$\Phi \Rightarrow$ The ~~Lor~~ simplest Lorentz invariant equation of motion:

$$\Box \phi(x,t) \equiv \boxed{\Box \phi(x^\mu) = 0}$$

→ Repeating this calculation with a potential term, say $m^2 \phi$:

$$\boxed{(\Box - m^2)\phi(x) = 0} \leftarrow \text{Klein-Gordon eq.}$$

<u>NOTE:</u> The above equation is classical !!

$\boxed{Q}$ So what is $\phi(x)$?

→ $\phi(x) = \phi(x^\mu) = \phi(\vec{x}, t)$ is called a "<u>field</u>".

→ $\phi(x)$ is a <u>scalar quantity</u> i.e it is invariant under Lorentz transformation.

$$\phi(x) \longrightarrow \boxed{\phi(\Lambda x) = \phi(x)}$$

→ $\phi(x)$ is known as the Klein-Gordon field.

→ Obvious questions:

[Q] What is the Hamiltonian/Lagrangian of $\phi(x)$?

[Q] Is $\phi(x)$ a wavefunction?

→ Let's start by answering the second question:

Consider the massless KG eq:

$$\Box\phi = 0$$

Let, $\phi(x) = a_p(t)\, e^{i\vec{p}\cdot\vec{x}}$

$$\Rightarrow \left(\partial_t^2 - \nabla^2\right) a_p(t)\, e^{i\vec{p}\cdot\vec{x}} = 0$$

$$\Rightarrow \left(\partial_t^2 + \vec{p}^{\,2}\right) a_p(t) = 0$$

← EoM of SHO!! But with $|\vec{p}|^2$ as parameter.

∴ For a given parameter $|\vec{p}|$,

KG eq. ⟺ SHO

⇒ For all $|\vec{p}|$: KG eq. ⟺ $\sum_{\text{all } \vec{p}} (\text{SHO})_p$

$$\Rightarrow \phi(x) = \int \frac{d^3p}{(2\pi)^3}\left[a_p(t)\, e^{i\vec{p}\cdot\vec{x}} + a_p^*(t)\, e^{-i\vec{p}\cdot\vec{x}}\right]$$

OR

$$\boxed{\phi(x) = \int \frac{d^3p}{(2\pi)^3} \left(a_p e^{-ip_\mu x^\mu} + a_p^* e^{ip_\mu x^\mu} \right)}$$

where,

$$p_\mu \equiv (\omega_{|\vec{p}|}, \vec{p}) \; ; \; \omega_p \equiv |\vec{p}|$$

& $$p_\mu x^\mu = \omega_p x^0 - \vec{p} \cdot \vec{x}$$

[Q] Does $\phi(x)$ look like a wavefunction?

→ Not quite!

→ a_p, a_p^* ~~tata~~ are upgraded to be the operators that create & annihilate states.

⇒ $\boxed{\phi(x) \text{ cannot be a wavefunction!}}$

→ Now, the first question:

A field appears when STR+CM is considered. It is convenient therefore to construct a Lagrangian first, since it is easier to write down the terms that comply with symmetries.

→ Note: Since $\phi = \phi(\vec{x}, t)$

We must write a Lagrangian density; such that:

$$\boxed{L = \int d^3x \, \mathcal{L}[\phi, \partial_\mu \phi]}$$

For a K-G field:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

→ Scalar !

→ ~~dim[$\mathcal{L}$] = ?~~

~~H/W (1) Variate $\mathcal{L}$ wrt $\phi(x)$.~~

~~(2) Find the EoM.~~

H/W (1) What is the dimension of $\mathcal{L}$?

(2) What is the dim[ϕ] ?

→ Deriving the EoM of K-G field:

"Euler-Lagrange eq."

Consider, $S = \int d^4x\, \mathcal{L}[\phi, \partial_\mu\phi]$

$$\Rightarrow \delta S = \int d^4x\, \delta\mathcal{L}[\phi, \partial_\mu\phi]$$

$$= \int d^4x \left[\frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right]$$

$$= \int d^4x \left(\left[\frac{\delta\mathcal{L}}{\delta\phi}\right] - \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right]\right)\delta\phi$$

($\because$ total derivatives don't contribute)

Least action principle:

$\delta S = 0$ for any $\delta\phi$

$$\Rightarrow \boxed{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0} \leftarrow \text{Euler-Lagrange eq.}$$

Applying the E-L eq. for K-G $\mathcal{L}$:

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2 \varphi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \partial^\mu \varphi$$

$$\Rightarrow \quad \boxed{\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0} \leftarrow \text{K-G eq.}$$

<u>NOTE:</u> This "Hamilton's principle" for fields constitutes the <u>Classical Field Theory</u>.

(Q) What is the field Hamiltonian? → for K-G field?? (H/W)

→ ‖er to L:

$$H = \int d^3x \, \underbrace{\mathcal{H}[\varphi, \pi]}_{\text{Hamiltonian density}}$$

& the Legendre transform of densities:

~~$\mathcal{H}[\varphi,\pi] = \pi[\varphi,\dot\varphi]\,\dot\varphi - \mathcal{H}[\varphi,\pi[$~~

$$\mathcal{H}[\varphi, \pi[\varphi,\dot\varphi]] = \pi[\varphi,\dot\varphi]\,\dot\varphi - \mathcal{L}[\varphi, \partial_\mu \varphi]$$

where,

$$\boxed{\pi[\varphi,\dot\varphi] \equiv \frac{\partial \mathcal{L}}{\partial \dot\varphi}} \leftarrow \boxed{\text{Conjugate momenta}}$$

$(\dot\varphi \equiv \partial_t \varphi)$