The Lorentz Group

The Lorentz transformations:

$$\left.\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\right\}$$ for frame O' moving with vel. $v\hat{x}$ wrt O.

Proof: (Recall the time dilation / length contraction)

Assuming $\Delta x = x - 0, \ldots$, $\Delta t = t - 0, \ldots$

$$\Rightarrow \quad x'^2 - c^2 t'^2 = x^2 - c^2 t^2$$

~~Let, the inertial frame is s.t.~~

$\Rightarrow$ The solutions of x' & ct' s.t. the above eq is satisfied is the Lorentz trans.!

$\boxed{\text{H/W}}$ Check by substituting for x', t' that inv. eq is satisfied.

→ These transformations are called "boost". (two frames sep. by const. vel. vel.)

In general, there are other transf. that we are familiar with recently: "rotation".

$$x \longrightarrow x\cos\theta + y\sin\theta$$
$$y \longrightarrow -x\sin\theta + y\cos\theta.$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Or, $\quad x_i \longrightarrow R_{ij} x_j$

* Einstein convention : $R_{ij} x_j = \sum_j R_{ij} x_j$

* Properties of R_{ij} :

$\rightarrow \quad R^T = R^{-1}$

$\rightarrow \quad R^T_{ij} R_{jk} = \delta_{ik} = \mathbb{1}_{ij}$

$\rightarrow \quad R_{ki} R_{lj} \delta_{kl} = (R^T R)_{ij} = \delta_{ij}$

$\rightarrow \quad x'^i x_i = (x^i R^T_{ij})(R_{jk} x_k) = x^i \delta_{ik} x_k = x^i x_i$

$\Rightarrow \boxed{R \text{ op. forms a "group".}}$

rotation group $[SO(n)]$ → dim.

$$\boxed{\text{Lorentz transformation} = \text{Rotation} + \text{Boost.}}$$

$$\Downarrow$$

$$\boxed{\text{Lorentz Group.}}$$

$$\therefore \quad R \longrightarrow \Lambda$$

i.e Lorentz "rotation" i.e. Λ preserves

$$s^2 = c^2t^2 - x^2 - y^2 - z^2.$$

Note: $c = 1$ from now on.

$$\therefore \quad x^i \longrightarrow x^\mu = (t, x, y, z)$$

$$x_i \longrightarrow x_\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$s^2 = t^2 - x^2 - y^2 - z^2 = (t \ x \ y \ z) \underbrace{\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}}_{\boxed{\equiv \text{ metric } \eta^\mu_{\ \mu\nu}}} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

(describes the geometry)

In Einstein notation:

$$\Rightarrow \boxed{\begin{aligned} s^2 &= \eta_{\mu\nu} x^\mu x^\nu = \eta^\mu_{\ \nu} x_\mu x^\nu \\ &= \eta^{\mu\nu} x_\mu x_\nu \end{aligned}}$$

A Lorentz transformation is given by:

$$\Lambda : \boxed{x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}}$$

From known L.T. for boost in $\hat{x}$:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & & 1 \end{pmatrix}$$

$\boxed{H/W}$ Find $\Lambda^{\mu}{}_{\nu}$ for rotation about x & boost along $\hat{x}$?

$\boxed{H/W}$ Find same for $\hat{z}$.?

* Scalar field: A fn. ~~f(x)~~ $\phi(x^{\mu})$ that is invariant under Λ trans.

* Vector field: A fn. $V^{\mu}(x^{\nu})$ that transforms as:

$$\boxed{V^{\mu} \longrightarrow \Lambda^{\mu}{}_{\nu} V^{\nu}}$$

* constructing ~~vector:~~ scalar:

$$V^{\mu} V_{\mu} = V^2 \leftarrow \text{scalar!}$$

* $\boxed{\Lambda^T g \Lambda = g} \leftarrow$ Lorentz group.