Notes qft canonical quantization

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$$(\Box + m^2)\phi = 0$$

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} \tilde{\phi}(p) e^{+ip\cdot x}$$

$$\Rightarrow (\Box + m^2)\int \frac{d^4p}{(2\pi)^4} \tilde{\phi}(p) e^{ip\cdot x} = 0$$

$$\Rightarrow \int \left(-p_\mu p^\mu + \cancel{m^2}\right) \tilde{\phi}(p) e^{ip\cdot x} \frac{d^4p}{(2\pi)^4} = 0$$

$$\Rightarrow \boxed{\int \frac{d^4p}{(2\pi)^4} \left[-\omega^2 + \vec{p}^{\,2}\right] \tilde{\phi}(p) e^{ip\cdot x} = 0}$$

$\Rightarrow \omega^2 = |\vec{p}|^2$

or

$\boxed{\omega^2 = |\vec{p}|^2 + m^2}$ (for massive ple.)

$\Rightarrow \boxed{\omega_p \equiv \omega = \sqrt{|\vec{p}|^2 + m^2}}$

$\Downarrow$

$a_{\vec{p}}, a^\dagger_{\vec{p}}$ always appears with factor of ω_p.

$$\Rightarrow \int \frac{d^3p}{(2\pi)^4} d\omega \left(-\omega^2 + \vec{p}^{\,2}\right) \tilde{\phi}(p) e^{\underbrace{ip\cdot x}_{+\omega t - \vec{p}\cdot\vec{x}}} = 0$$

$$\Rightarrow \int \frac{d^3p}{(2\pi)^{3+1}} \left[\frac{\partial^2}{\partial t^2} + \vec{p}^{\,2}\right] \underbrace{\left(\int \frac{d\omega}{2\pi} e^{i\omega t} \tilde{\phi}(p)\right)}_{a_{\vec{p}}(t)} \cdot e^{-i\vec{p}\cdot\vec{x}} = 0$$

Now, for each $\vec{p}$:

$$\left(\frac{\partial^2}{\partial t^2} + |\vec{p}|^2\right) a_{\vec{p}}(t) = 0 \leftarrow \text{SHO!!}$$

where $a_{\vec{p}}(t)$ is defined as:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} a_{\vec{p}}(t) + \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} a^*_{\vec{p}}(t)$$

Quantization

→ Promote $\underbrace{a_p, a_p^*}$ to operators.

$\downarrow$

$a_{\vec{p}}, a^\dagger_{\vec{p}}$ with $[a_{\vec{p}}, a^\dagger_{\vec{p}}] = 1$

→ But, $\exists\ \vec{p} \in \mathbb{R}^3 \Rightarrow$ infinite # of $\vec{p}$!!

$$\Rightarrow \int [a_{\vec{p}}, a^\dagger_{\vec{p}}] \frac{d^3p}{(2\pi)^3} = 1$$

$$\Rightarrow [a_{\vec{p}}, a^\dagger_{\vec{p}'}] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,')$$

→ Defining vacuum: $|0\rangle$ such that,

$$a^\dagger_{\vec{p}}|0\rangle = \frac{1}{\sqrt{2\omega_p}}|\vec{p}\rangle$$

$$a_{\vec{p}}|0\rangle = 0$$

Δ $\boxed{\langle 0|0\rangle = 1}$

→ Inner product of kets:

$$\langle p|k\rangle = \langle 0|a_p a_k^\dagger|0\rangle$$

$$= 2\omega_p (2\pi)^3 \delta^3(\vec{p} - \vec{k})$$

→ Completeness relation:

$$\mathbb{1} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{|\vec{p}|}} |\vec{p}\rangle\langle\vec{p}|$$