where the subscript 0 indicates this is a free field. The factor of $\sqrt{2\omega_p}$ is included for later convenience.

This equation looks just like the classical free-particle solutions, Eq. (2.59), to Maxwell's equations (ignoring polarizations) but instead of a_p and $a_p^\dagger$ being *functions*, they are now the annihilation and creation *operators* for that mode. Sometimes we say the classical a_p is c-number valued and the quantum one is q-number valued. The connection with Eq. (2.59) is only suggestive. The quantum equation, Eq. (2.75), should be taken as the definition of a field operator $\phi_0(\vec{x})$ constructed from the creation and annihilation operators a_p and $a_p^\dagger$.

To get a sense of what the operator ϕ_0 does, we can act with it on the vacuum and project out a momentum component:

$$\begin{aligned}\langle\vec{p}|\phi_0(\vec{x})|0\rangle &= \langle 0|\sqrt{2\omega_p}a_p \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(a_k e^{i\vec{k}\vec{x}} + a_k^\dagger e^{-i\vec{k}\vec{x}}\right)|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_p}{\omega_k}}\left[e^{i\vec{k}\vec{x}}\langle 0|a_p a_k|0\rangle + e^{-i\vec{k}\vec{x}}\langle 0|a_p a_k^\dagger|0\rangle\right] \\ &= e^{-i\vec{p}\vec{x}}. \end{aligned} \tag{2.76}$$

This is the same thing as the projection of a position state on a momentum state in one-particle quantum mechanics:

$$\langle\vec{p}|\vec{x}\rangle = e^{-i\vec{p}\vec{x}}. \tag{2.77}$$

So, $\phi_0(\vec{x})|0\rangle = |\vec{x}\rangle$, that is, $\phi_0(\vec{x})$ creates a particle at position $\vec{x}$. This should not be surprising, since $\phi_0(x)$ in Eq. (2.75) is very similar to $x = a + a^\dagger$ in the simple harmonic oscillator. Since ϕ_0 is Hermitian, $\langle\vec{x}| = \langle 0|\phi_0(\vec{x})$ as well.

By the way, there are many states $|\psi\rangle$ in the Fock space that satisfy $\langle\vec{p}|\psi\rangle = e^{-i\vec{p}\vec{x}}$. Since $\langle\vec{p}|$ only has non-zero matrix elements with one-particle states, adding to $|x\rangle$ a two- or zero-particle state, as in $\phi_0^2(\vec{x})|0\rangle$, has no effect on $\langle\vec{p}|\vec{x}\rangle$. That is, $|\psi\rangle = \left(\phi_0(\vec{x}) + \phi_0^2(\vec{x})\right)|0\rangle$ also satisfies $\langle\vec{p}|\psi\rangle = e^{-i\vec{p}\vec{x}}$. The state $|\vec{x}\rangle \equiv \phi_0(\vec{x})|0\rangle$ is the unique *one-particle* state with $\langle\vec{p}|\psi\rangle = e^{-i\vec{p}\vec{x}}$.

2.3.2 Time dependence

In quantum field theory, we generally work in the Heisenberg picture, where all the time dependence is in operators such as ϕ and a_p. For free fields, the creation and annihilation operators for each momentum $\vec{p}$ in the quantum field are just those of a simple harmonic oscillator. These operators should satisfy Eq. (2.55), $a_p(t) = e^{-i\omega_p t}a_p$, and its conjugate $a_p^\dagger(t) = e^{i\omega_p t}a_p^\dagger$, where a_p and $a_p^\dagger$ (without an argument) are time independent. Then, we can *define* a quantum scalar field as

$$\phi_0(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{-ipx} + a_p^\dagger e^{ipx}\right), \tag{2.78}$$

with $p^\mu \equiv (\omega_p, \vec{p})$ and $\omega_p = |\vec{p}|$ as in Eq. (2.60). The 0 subscript still indicates that these are free fields.

To be clear, there is no physical content in Eq. (2.78). It is just a definition. The physical content is in the algebra of a_p and $a_p^\dagger$ and in the Hamiltonian H_0. Nevertheless, we will see that collections of a_p and $a_p^\dagger$ in the form of Eq. (2.78) are very useful in quantum field theory. For example, you may note that while the integral is over only three components of p_μ, the phases have combined into a manifestly Lorentz-invariant form. This field now automatically satisfies $\Box\phi(x) = 0$. If a scalar field had mass m, we could still write it in exactly the same way but with a massive dispersion relation: $\omega_p \equiv \sqrt{\vec{p}^2 + m^2}$. Then the quantum field still satisfies the classical equation of motion: $(\Box + m^2)\phi(x) = 0$.

Let us check that our free Hamiltonian is consistent with the expectation for time evolution. Commuting the free fields with H_0 we find

$$\begin{aligned}[H_0, \phi_0(\vec{x}, t)] &= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left[\omega_p \left(a_p^\dagger a_p + \frac{1}{2}\right), a_k e^{-ikx} + a_k^\dagger e^{ikx}\right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[-\omega_p a_p\, e^{-ipx} + \omega_p a_p^\dagger e^{ipx}\right] \\ &= -i\partial_t \phi_0(\vec{x}, t), \end{aligned} \tag{2.79}$$

which is exactly the expected result.

For any Hamiltonian, quantum fields satisfy the Heisenberg equations of motion:

$$i\partial_t \phi(x) = [\phi, H]. \tag{2.80}$$

In a free theory, $H = H_0$, and this is consistent with Eq. (2.78). In an interacting theory, that is, one whose Hamiltonian H differs from the free Hamiltonian H_0, the Heisenberg equations of motion are still satisfied, but we will rarely be able to solve them exactly. To study interacting theories, it is often useful to use the same notation for interacting fields as for free fields:

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p(t) e^{-ipx} + a_p^\dagger(t) e^{ipx}\right]. \tag{2.81}$$

At any *fixed time*, the full interacting creation and annihilation operators $a_p^\dagger(t)$ and $a_p(t)$ satisfy the same algebra as in the free theory – the Fock space is the same at every time, due to time-translation invariance. We can therefore define the exact creation operators $a_p(t)$ to be equal to the free creation operators a_p at any given fixed time, $a_p(t_0) = a_p$ and so $\phi(\vec{x}, t_0) = \phi_0(\vec{x}, t_0)$. However, the operators that create particular momentum states $|p\rangle$ in the interacting theory mix with each other as time evolves. We generally will not be able to solve the dynamics of an interacting theory exactly. Instead, we will expand $H = H_0 + H_{\text{int}}$ and calculate amplitudes using time-dependent perturbation theory with H_{int}, just as in quantum mechanics. In Chapter 7, we use this approach to derive the Feynman rules.

The first-quantized (quantum mechanics) limit of the second-quantized theory (quantum field theory) comes from restricting to the one-particle states, which is appropriate in the non-relativistic limit. A basis of these states is given by the vectors $\langle x| = \langle \vec{x}, t|$:

$$\langle x| = \langle 0|\, \phi(\vec{x}, t). \tag{2.82}$$

Then, a Schrödinger picture wavefunction is

$$\psi(x) = \langle x|\psi\rangle, \tag{2.83}$$

which satisfies

$$i\partial_t\psi(x) = i\partial_t\langle 0|\phi(\vec{x},t)|\psi\rangle = i\,\langle 0|\partial_t\phi(\vec{x},t)|\psi\rangle\,. \tag{2.84}$$

In the massive case, the free quantum field $\phi_0(x)$ satisfies $\partial_t^2\phi_0 = \left(\vec{\nabla}^2 - m^2\right)\phi_0$ and we have from Eq. (2.79) (with the massive dispersion relation $\omega_p = \sqrt{\vec{p}^2 + m^2}$):

$$\begin{aligned} i\langle 0|\partial_t\phi(\vec{x},t)|\psi\rangle &= \langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{\sqrt{\vec{p}^2+m^2}}{\sqrt{2\omega_p}}\left(a_p e^{-ipx} - a_p^\dagger e^{ipx}\right)|\psi\rangle \\ &= \langle 0|\sqrt{m^2 - \vec{\nabla}^2}\phi_0(x)|\psi\rangle\,. \end{aligned} \tag{2.85}$$

So,

$$i\partial_t\psi(x) = \sqrt{m^2 - \vec{\nabla}^2}\psi(x) = \left(m - \frac{\vec{\nabla}^2}{2m} + \mathcal{O}\left(\frac{1}{m^2}\right)\right)\psi(x). \tag{2.86}$$

The final form is the low-energy (large-mass) expansion. We can then define the non-relativistic Hamiltonian by subtracting off the mc^2 contribution to the energy, which is irrelevant in the non-relativistic limit. This gives

$$i\partial_t\psi(x) = -\frac{\vec{\nabla}^2}{2m}\psi(x), \tag{2.87}$$

which is the non-relativistic Schrödinger equation for a free theory. Another way to derive the quantum mechanics limit of quantum field theory is discussed in Section 33.6.2.

2.3.3 Commutation relations

We will occasionally need to use the equal-time commutation relations of the second-quantized field and its time derivative. The commutator of a field at two different points is

$$\begin{aligned} [\phi(\vec{x}),\phi(\vec{y})] &= \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p 2\omega_q}}\left[\left(a_p e^{i\vec{p}\vec{x}} + a_p^\dagger e^{-i\vec{p}\vec{x}}\right),\left(a_q e^{i\vec{q}\vec{y}} + a_q^\dagger e^{-i\vec{q}\vec{y}}\right)\right] \\ &= \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p 2\omega_q}}\left(e^{i\vec{p}\vec{x}}e^{-i\vec{q}\vec{y}}\left[a_p,a_q^\dagger\right] + e^{-i\vec{p}\vec{x}}e^{i\vec{q}\vec{y}}\left[a_p^\dagger,a_q\right]\right). \end{aligned} \tag{2.88}$$

Using Eq. (2.69), $[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\vec{p}-\vec{k})$, this becomes

$$\begin{aligned} [\phi(\vec{x}),\phi(\vec{y})] &= \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p 2\omega_q}}\left[e^{i\vec{p}\vec{x}}e^{-i\vec{q}\vec{y}} - e^{-i\vec{p}\vec{x}}e^{i\vec{q}\vec{y}}\right](2\pi)^3\,\delta^3(\vec{p}-\vec{q}) \\ &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left[e^{i\vec{p}(\vec{x}-\vec{y})} - e^{-i\vec{p}(\vec{x}-\vec{y})}\right]. \end{aligned} \tag{2.89}$$

Since the integral measure and $\omega_p = \sqrt{\vec{p}^2 + m^2}$ are symmetric under $\vec{p} \to -\vec{p}$ we can flip the sign on the exponent of one of the terms to see that the commutator vanishes:

$$[\phi(\vec{x}), \phi(\vec{y})] = 0. \tag{2.90}$$

The equivalent calculation at different times is much more subtle (we discuss the general result in Section 12.6 in the context of the spin-statistics theorem).

Next, we note that the time derivative of the free field, at $t = 0$, has the form

$$\pi(\vec{x}) \equiv \partial_t \phi(x)\Big|_{t=0} = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} \left(a_p e^{i\vec{p}\vec{x}} - a_p^\dagger e^{-i\vec{p}\vec{x}}\right), \tag{2.91}$$

where π is the operator canonically conjugate to ϕ. As $\phi(\vec{x})$ is the second-quantized analog of the $\hat{x}$ operator, $\pi(\vec{x})$ is the analog of the $\hat{p}$ operator. Note that $\pi(\vec{x})$ has nothing to do with the physical momentum of states in the Hilbert space: $\pi(\vec{x})\,|0\rangle$ is not a state of given momentum. Instead, it is a state also at position $\vec{x}$ created by the time derivative of $\phi(\vec{x})$.

Now we compute

$$\begin{aligned}
[\phi(\vec{x}), \pi(\vec{y})] &= -i \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} \frac{1}{\sqrt{2\omega_q}} \left(e^{i\vec{p}\vec{y}} e^{-i\vec{q}\vec{x}} \left[a_q^\dagger, a_p\right] - e^{i\vec{q}\vec{x}} e^{-i\vec{p}\vec{y}} \left[a_q, a_p^\dagger\right]\right) \\
&= \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left[e^{i\vec{p}(\vec{x}-\vec{y})} + e^{-i\vec{p}(\vec{x}-\vec{y})}\right].
\end{aligned} \tag{2.92}$$

Both of these integrals give $\delta^3(\vec{x} - \vec{y})$, so we find

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x} - \vec{y}), \tag{2.93}$$

which is the analog of $[\hat{x}, \hat{p}] = i$ in quantum mechanics. It encapsulates the field theory version of the uncertainty principle: you cannot know the properties of the field and its rate of change at the same place at the same time.

In a general interacting theory, at any fixed time, $\phi(\vec{x})$ and $\pi(\vec{x})$ have expressions in terms of creation and annihilation operators whose algebra is identical to that of the free theory. Therefore, they satisfy the commutation relations in Eqs. (2.90) and (2.93) as well as $[\pi(\vec{x}), \pi(\vec{y})] = 0$. The Hamiltonian in an interacting theory should be expressed as a functional of the operators $\phi(\vec{x})$ and $\pi(\vec{x})$ with time evolution given by $\partial_t \mathcal{O} = i[H, \mathcal{O}]$. Any such Hamiltonian can then be expressed entirely in terms of creation and annihilation operators using Eqs. (2.75) and (2.91); thus it has a well-defined action on the associated Fock space. Conversely, it is sometimes more convenient (especially for non-relativistic or condensed matter applications) to derive the form of the Hamiltonian in terms of a_p and $a_p^\dagger$. We can then express a_p and $a_p^\dagger$ in terms of $\phi(\vec{x})$ and $\pi(\vec{x})$ by inverting Eqs. (2.75) and (2.91) for a_p and $a_p^\dagger$ (the solution is the field theory equivalent of Eq. (2.47)).

In summary, all we have done to quantize the electromagnetic field is to treat it as an infinite set of simple harmonic oscillators, one for each wavenumber $\vec{p}$. More generally:

Quantum field theory is just quantum mechanics with an infinite number of harmonic oscillators.