These are block diagonal. These are the same generators we used for the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations above. This makes it clear that the Dirac representation of the Lorentz group is reducible; it is the direct sum of a left-handed and a right-handed spinor representation.

Another representation is the **Majorana representation**:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}. \tag{10.74}$$

In this basis the γ-matrices are purely imaginary. The Majorana is another $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz group that is physically equivalent to the Weyl representation.

The Weyl spinors, ψ_L and ψ_R, are in a way more fundamental than Dirac spinors such as ψ because they correspond to irreducible representations of the Lorentz group. But the electron is a Dirac spinor. More importantly, QED is symmetric under $L \leftrightarrow R$. Thus, for QED the γ-matrices make calculations a lot easier than separating out the ψ_L and ψ_R components. In fact, we will develop such efficient machinery for manipulating the γ-matrices that even in theories which are not symmetric to $L \leftrightarrow R$, such as the theory of weak interactions (Chapter 29), it will be convenient to embed the Weyl spinors into Dirac spinors and add projectors to remove the unphysical components. These projections are discussed in Section 11.1.

10.3.1 Lorentz transformation properties

When using Dirac matrices and spinors, we often suppress spinor indices but leave vector indices explicit. So an equation such as $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ really means

$$\gamma^\mu_{\alpha\gamma}\gamma^\nu_{\gamma\beta} + \gamma^\nu_{\alpha\gamma}\gamma^\mu_{\gamma\beta} = 2g^{\mu\nu}\delta^{\alpha\beta}, \tag{10.75}$$

and the equation $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$ means

$$S^{\mu\nu}_{\alpha\beta} = \frac{i}{4}\left(\gamma^\mu_{\alpha\gamma}\gamma^\nu_{\gamma\beta} - \gamma^\nu_{\alpha\gamma}\gamma^\mu_{\gamma\beta}\right). \tag{10.76}$$

For an expression such as

$$V^2 = V_\mu g^{\mu\nu} V_\nu = \frac{1}{2} V_\mu \{\gamma^\mu,\gamma^\nu\} V_\nu \tag{10.77}$$

to be invariant, the Lorentz transformations in the vector and Dirac representations must be related. Indeed, since $\bar{\psi}\gamma^\mu\psi$ transforms like a 4-vector we can deduce that

$$\Lambda_s^{-1}\gamma^\mu\Lambda_s = (\Lambda_V)^{\mu\nu}\gamma^\nu, \tag{10.78}$$

where the Λ_s are the Lorentz transformations acting on spinor indices and Λ_V are the Lorentz transformations in the vector representation. Writing out the matrix indices $\gamma^{\alpha\beta}_\mu$ this means

$$(\Lambda_s^{-1})_{\delta\alpha}\gamma^\mu_{\alpha\beta}(\Lambda_s)_{\beta\gamma} = (\Lambda_V)^{\mu\nu}\gamma^\nu_{\delta\gamma}, \tag{10.79}$$

where μ refers to which γ-matrix, and α and β index the elements of that matrix. You can check this with the explicit forms for Λ_V and Λ_s in Eqs. (10.70) and (10.71) above.

It is useful to study the properties of the Lorentz generators from the Dirac algebra itself, without needing to choose a particular basis for the γ^μ. First note that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad \Rightarrow \quad (\gamma^0)^2 = \mathbb{1}, \quad (\gamma^i)^2 = -\mathbb{1}. \tag{10.80}$$

So the eigenvalues of γ^0 are ± 1 and the eigenvalues of γ^i are $\pm i$. Thus, if we diagonalize γ^0, we will see that it is Hermitian, and if we diagonalize γ^1, γ^2 or γ^3 we will see that they are anti-Hermitian. This is true, in general, for any representation of the γ-matrices:

$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{i\dagger} = -\gamma^i. \tag{10.81}$$

Then,

$$(S^{\mu\nu})^\dagger = \left(\frac{i}{4}[\gamma^\mu, \gamma^\nu]\right)^\dagger = -\frac{i}{4}\left[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}\right] = \frac{i}{4}\left[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}\right], \tag{10.82}$$

which implies

$$S^{ij\dagger} = S^{ij}, \quad S^{0i\dagger} = -S^{0i}. \tag{10.83}$$

Again, we see that the rotations are unitary and the boosts are not. You can see this from the explicit representations in Eq (10.73). But because we showed it algebraically, using only the defining equation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, it is true in *any* representation of the Dirac algebra.

Now, observe that *one* of the Dirac matrices is Hermitian, γ^0. Moreover,

$$\gamma^0\gamma^i\gamma^0 = -\gamma^i = \gamma^{i\dagger}, \quad \gamma^0\gamma^0\gamma^0 = \gamma^0 = \gamma^{0\dagger}, \tag{10.84}$$

so $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$. Then

$$\gamma^0 (S^{\mu\nu})^\dagger \gamma^0 = \gamma^0 \frac{i}{4}\left[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}\right]\gamma^0 = \frac{i}{4}\left[\gamma^0\gamma^{\mu\dagger}\gamma^0, \gamma^0\gamma^{\nu\dagger}\gamma^0\right] = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = S^{\mu\nu}, \tag{10.85}$$

and so

$$(\gamma^0\Lambda_s\gamma^0)^\dagger = \gamma^0 \exp(i\theta_{\mu\nu}S^{\mu\nu})^\dagger\gamma^0 = \exp(-i\theta_{\mu\nu}\gamma^0 S^{\mu\nu\dagger}\gamma^0) = \exp(-i\theta_{\mu\nu}S^{\mu\nu}) = \Lambda_s^{-1}. \tag{10.86}$$

Then, finally,

$$\psi^\dagger\gamma^0\psi \to (\psi^\dagger\Lambda_s^\dagger)\gamma^0(\Lambda_s\psi) = (\psi^\dagger\gamma^0\Lambda_s^{-1}\Lambda_s\psi) = \psi^\dagger\gamma^0\psi, \tag{10.87}$$

which is Lorentz invariant.

We have just been re-deriving from the Dirac algebra point of view what we found by hand from the Weyl point of view. We have seen that the natural conjugate for ψ out of which real Lorentz-invariant expressions are constructed is not $\psi^\dagger$ but

$$\bar{\psi} \equiv \psi^\dagger\gamma^0. \tag{10.88}$$

The point is that $\bar{\psi}$ transforms according to Λ_s^{-1}. Thus $\bar{\psi}\psi$ is Lorentz invariant. In contrast, $\psi^\dagger\psi$ is not Lorentz invariant, since $\psi^\dagger\psi \to (\psi^\dagger\Lambda_s^\dagger)(\Lambda_s\psi)$. For this to be invariant, we would need $\Lambda_s^\dagger = \Lambda_s^{-1}$, that is, for the representation of the Lorentz group to be unitary.

But the finite-dimensional spinor representation of the Lorentz group, like the 4-vector representation, is *not* unitary, because the boost generators are anti-Hermitian. As with vectors, for unitary representations we will need *fields* $\psi(x)$ that transform in infinite-dimensional representations of the Poincaré group.

We can also construct objects such as

$$\bar{\psi}\gamma_\mu\psi, \quad \bar{\psi}\gamma_\mu\gamma_\nu\psi, \quad \bar{\psi}\partial_\mu\psi; \tag{10.89}$$

all transform like tensors under the Lorentz group. Also

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{10.90}$$

is Lorentz invariant. We abbreviate this with

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \tag{10.91}$$

which is the Dirac Lagrangian.

The Dirac equation follows from this Lagrangian by the equations of motion:

$$(i\partial\!\!\!/ - m)\psi = 0. \tag{10.92}$$

To be explicit, this is shorthand for

$$(i\gamma^\mu_{\alpha\beta}\partial_\mu - m\delta_{\alpha\beta})\psi_\beta = 0. \tag{10.93}$$

After multiplying the Dirac equation by $(i\partial\!\!\!/ + m)$ we find

$$\begin{aligned} 0 = (i\partial\!\!\!/ + m)(i\partial\!\!\!/ - m)\psi &= \left(-\frac{1}{2}\partial_\mu\partial_\nu\{\gamma^\mu, \gamma^\nu\} - \frac{1}{2}\partial_\mu\partial_\nu[\gamma^\mu, \gamma^\nu] - m^2\right)\psi \\ &= -(\partial^2 + m^2)\psi. \end{aligned} \tag{10.94}$$

So ψ satisfies the Klein–Gordon equation:

$$(\Box + m^2)\psi = 0. \tag{10.95}$$

In Fourier space, this implies that on-shell spinor momenta satisfy the unique relativistic dispersion relation $p^2 = m^2$, just like scalars. Because spinors also satisfy an equation linear in derivatives, people sometimes say the Dirac equation is the "square root" of the Klein–Gordon equation.

We can integrate the Lagrangian by parts to derive the equations of motion for $\bar{\psi}$:

$$\mathcal{L} = \bar{\psi}i\partial\!\!\!/\psi - m\bar{\psi}\psi = -i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi - m\bar{\psi}\psi. \tag{10.96}$$

So,

$$-i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} = 0. \tag{10.97}$$

This γ^μ on the opposite side from ∂_μ is a little annoying, so we often write

$$\bar{\psi}(-i\overleftarrow{\partial\!\!\!/} - m) = 0, \tag{10.98}$$

where the derivative acts to the left. This makes the conjugate equation look more like the original Dirac equation.