specifying their states, such as momentum, spin, etc. We are concerned only with nonrelativistic, elastic scattering of structureless spinless particles.

In the next three sections, we deal with a formalism that describes a single particle scattering from a potential $V(\mathbf{r})$. As it stands, the formalism describes a particle colliding with an immobile target whose only role is to provide the potential. (This picture provides a good approximation to processes where a light particle collides with a very heavy one, say an α particle colliding with a heavy nucleus.) In Section 19.6 we see how, upon proper interpretation, the same formalism describes two-body collisions in the CM frame. In that section we will also see how the description of the scattering process in the CM frame can be translated to another frame, called the lab frame, where the target is initially at rest. It is important to know how to pass from one frame to the other, since theoretical calculations are most easily done in the CM frame, whereas most experiments are done in the lab frame.

19.2. Recapitulation of One-Dimensional Scattering and Overview

Although we are concerned here with scattering in three dimensions, we begin by recalling one-dimensional scattering, for it shares many common features with its three-dimensional counterpart. The practical question one asks is the following: If a beam of nearly monoenergetic particles with mean momenta $\langle P\rangle=\hbar k_0$ are incident from the far left ($x\to-\infty$) on a potential $V(x)$ which tends to zero as $|x|\to\infty$, what fraction T will get transmitted and what fraction R will get reflected?‡ It is not *a priori* obvious that the above question can be answered, since the mean momentum does not specify the quantum states of the incoming particles. But it turns out that if the individual momentum space wave functions are sharply peaked at $\hbar k_0$, the reflection and transmission probabilities depend only on k_0 and not on the detailed shapes of the wave functions. Thus it is possible to calculate $R(k_0)$ and $T(k_0)$ that apply to every particle in the beam. Let us recall some of the details.

(1) We start with some wave packet, say a Gaussian, with $\langle P\rangle=\hbar k_0$ and $\langle X\rangle\to-\infty$.

(2) We expand this packet in terms of the eigenfunctions ψ_k of $H=T+V$ with coefficients $a(k)$. The functions ψ_k have the following property:

$$\begin{aligned}\psi_k &\xrightarrow[x\to-\infty]{} A\,e^{-ikx}+B\,e^{ikx}\\ &\xrightarrow[x\to\infty]{} C\,e^{ikx}\end{aligned}\tag{19.2.1}$$

In other words, the asymptotic form of ψ_k contains an incident wave $A\,e^{ikx}$ and a reflected wave $B\,e^{-ikx}$ as $x\to-\infty$, and just a transmitted wave $C\,e^{ikx}$ as $x\to\infty$. Although the most general solution also contains a $D\,e^{-ikx}$ piece as $x\to\infty$, we set

‡ In general, the particle can come in from the far right as well. Also $V(x)$ need not tend to zero at both ends, but to constants V_+ and V_- as $x\to\pm\infty$. We assume $V_+=V_-=0$ for simplicity. We also assume $|xV(x)|\to 0$ as $|x|\to\infty$, so that the particle is asymptotically free ($\psi\sim e^{\pm ikx}$).

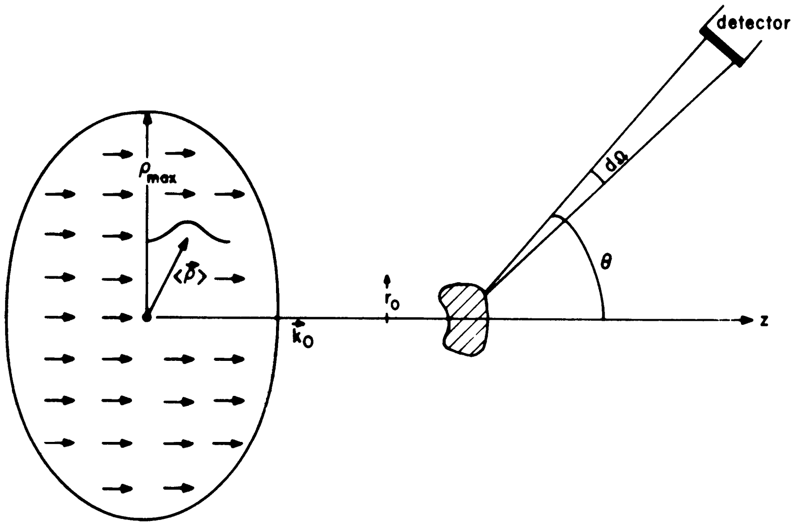

Figure 19.1. A schematic description of scattering. The incident particles, shown by arrows, are really described by wave packets (only one is shown) with mean momentum $\langle \mathbf{P} \rangle = \langle \hbar \mathbf{k}_0 \rangle$ and mean impact parameter $\langle \boldsymbol{\rho} \rangle$ uniformly distributed in the $\boldsymbol{\rho}$-plane out to $\boldsymbol{\rho}_{\max} \gg r_0$, the range of the potential. The shaded region near the origin stands for the domain where the potential is effective. The detector catches all particles that emerge in the cone of opening angle $d\Omega$. The beam is assumed to be coming in along the z axis.

$D=0$ on physical grounds: the incident wave $A\,e^{ikx}$ can only produce a right-going wave as $x \to \infty$.

(3) We propagate the wave packet in time by attaching to the expansion coefficients $a(k)$ the time dependence $e^{-iEt/\hbar}$, where $E = \hbar^2 k^2/2\mu$. We examine the resulting solution as $t \to \infty$ and identify the reflected and transmitted packets. From the norms of these we get R and T respectively.

(4) We find at this stage that if the incident packet is sharply peaked in momentum space at $\hbar k_0$, R and T depend only on k_0 and not on the detailed shape of the wave function. Thus the answer to the question raised at the outset is that a fraction $R(k_0)$ of the incident particles will get reflected and a fraction $T(k_0)$ will get transmitted.

(5) Having done all this hard work, we find at the end that the same result could have been obtained by considering just one eigenfunction ψ_{k_0} and taking the ratios of the transmitted and reflected current densities to the incident curent density.

The scattering problem in three dimensions has many similarities with its one-dimensional counterpart and also several differences that inevitably accompany the increase in dimensionality. First of all, the incident particles (coming out of the accelerator) are characterized, not by just the mean momentum $\langle \mathbf{P} \rangle = \hbar \mathbf{k}_0$, but also by the fact that they are uniformly distributed in the *impact parameter* $\boldsymbol{\rho}$, which is the coordinate in the plane perpendicular to $\mathbf{k}_0$ (Fig. 19.1). The distribution is of course not uniform out to $\rho \to \infty$, but only up to $\rho_{\max} \gg r_0$, where r_0, the *range of the potential*, is the distance scale beyond which the potential is negligible. [For instance, if $V(r) = e^{-r^2/a^2}$, the range $r_0 \simeq a$.] The problem is to calculate the rate at which particles get scattered into a far away detector that subtends a solid angle $d\Omega$ in the direction (θ, ϕ) measured relative to the beam direction (Fig. 19.1). To be precise, one wants the *differential cross section* $d\sigma/d\Omega$ defined as follows:

$$\frac{d\sigma(\theta,\phi)}{d\Omega}d\Omega=\frac{\text{number of particles scattered into } d\Omega/\text{sec}}{\text{number incident/sec/area in the } \boldsymbol{\rho} \text{ plane}} \tag{19.2.2}$$

The calculation of $d\sigma/d\Omega$ proceeds as follows.‡

(1) One takes some initial wave packet with mean momentum $\langle \mathbf{P}\rangle=\hbar\mathbf{k}_0$ and mean impact parameter $\langle \boldsymbol{\rho}\rangle$. The mean coordinate in the beam direction is not relevant, as long as it is far away from the origin.

(2) One expands the wave packet in terms of the eigenfunctions $\psi_{\mathbf{k}}$ of $H=T+V$ which are of the form

$$\psi_{\mathbf{k}}=\psi_{\text{inc}}+\psi_{\text{sc}} \tag{19.2.3}$$

where ψ_{inc} is the incident wave $e^{i\mathbf{k}\cdot\mathbf{r}}$ and ψ_{sc} is the scattered wave. One takes only those solutions in which ψ_{sc} is purely outgoing. We shall have more to say about ψ_{sc} in a moment.

(3) One propagates the wave packet by attaching the time-dependence factor $e^{-iEt/\hbar}$ $(E=\hbar^2k^2/2\mu)$ to each coefficient $a(\mathbf{k})$ in the expansion.

(4) One identifies the scattered wave as $t\to\infty$, and calculates the probability current density associated with it. One integrates the total flow of probability into the cone $d\Omega$ at (θ,ϕ). This gives the probability that the incident particle goes into the detector at (θ,ϕ). One finds that if the momentum space wave function of the incident wave packet is sharply peaked at $\langle \mathbf{P}\rangle=\hbar\mathbf{k}_0$, the probability of going into $d\Omega$ depends only on $\hbar\mathbf{k}_0$ and $\langle\boldsymbol{\rho}\rangle$. Call this probability $P(\boldsymbol{\rho},\mathbf{k}_0\to d\Omega)$.

(5) One considers next a beam of particle with $\eta(\boldsymbol{\rho})$ particles per second per unit area in the $\boldsymbol{\rho}$ plane. The number scattering into $d\Omega$ per second is

$$\eta(d\Omega)=\int P(\boldsymbol{\rho},\mathbf{k}_0\to d\Omega)\eta(\boldsymbol{\rho})\,d^2\boldsymbol{\rho} \tag{19.2.4}$$

Since in the experiment $\eta(\boldsymbol{\rho})=\eta$, a constant, we have from Eq. (192.2)

$$\frac{d\sigma}{d\Omega}d\Omega=\frac{\eta(d\Omega)}{\eta}=\int P(\boldsymbol{\rho},\mathbf{k}_0\to d\Omega)\,d^2\boldsymbol{\rho} \tag{19.2.5}$$

(6) After all this work is done one finds that $d\sigma/d\Omega$ could have been calculated from considering just the static solution $\psi_{\mathbf{k}_0}$ and computing in the limit $r\to\infty$, the ratio of the probability flow per second into $d\Omega$ associated with ψ_{sc}, to the incident probability current density associated with $e^{i\mathbf{k}_0\cdot\mathbf{r}}$. The reason the time-dependent picture reduces to the time-independent picture is the same as in one dimension: as we broaden the incident wave packet more and more in coordinate space, the incident and scattered waves begin to coexist in a steady-state configuration, $\psi_{\mathbf{k}_0}$. What about

‡ We do not consider the details here, for they are quite similar to the one-dimensional case. The few differences alone are discussed. See Taylor's book for the details.

the average over $\langle \boldsymbol{\rho} \rangle$? This is no longer necessary, since the incident packet is now a plane wave $e^{i\mathbf{k}_0 \cdot \mathbf{r}}$ which is already uniform in $\boldsymbol{\rho}$.‡

Let us consider some details of extracting $d\sigma/d\Omega$ from $\psi_{\mathbf{k}_0}$. Choosing the z axis parallel to $\mathbf{k}_0$ and dropping the subscript 0, we obtain

$$\psi_{\mathbf{k}} = e^{ikz} + \psi_{sc}(r, \theta, \phi) \tag{19.2.6}$$

where θ and ϕ are defined in Fig. 19.1. Although the detailed form of ψ_{sc} depends on the potential, we know that far from the origin it satisfies the free-particle equation [assuming $rV(r) \to 0$ as $r \to \infty$].

$$(\nabla^2 + k^2)\psi_{sc} = 0 \qquad (r \to \infty) \tag{19.2.7}$$

and is purely outgoing.

Recalling the general solution to the free-particle equation (in a region that excludes the origin) we get

$$\psi_{sc} \xrightarrow[r\to\infty]{} \sum_l \sum_m (A_l j_l(kr) + B_l n_l(kr)) Y_l^m(\theta, \phi) \tag{19.2.8}$$

Notice that we do not exclude the Neumann functions because they are perfectly well behaved as $r \to \infty$. Since

$$\begin{aligned} j_l(kr) &\xrightarrow[r\to\infty]{} \sin(kr - l\pi/2)/(kr) \\ n_l(kr) &\xrightarrow[r\to\infty]{} -\cos(kr - l\pi/2)/(kr) \end{aligned} \tag{19.2.9}$$

it must be that $A_l/B_l = -i$, so that we get a purely outgoing wave e^{ikr}/kr. With this condition, the asymptotic form of the scattered wave is

$$\psi_{sc} \xrightarrow[r\to\infty]{} \frac{e^{ikr}}{kr} \sum_l \sum_m (-i)^l (-B_l) Y_l^m(\theta, \phi) \tag{19.2.10}$$

or

$$\psi_{sc} \xrightarrow[r\to\infty]{} \frac{e^{ikr}}{r} f(\theta, \phi)§ \tag{19.2.11}$$

and

$$\psi_{\mathbf{k}} \xrightarrow[r\to\infty]{} e^{ikz} + f(\theta, \phi)\frac{e^{ikr}}{r} \tag{19.2.12}$$

where f is called the *scattering amplitude*.

‡ Let us note. as we did in one dimension, that a wave packet does not simply become a plane wave as we broaden it, for the former has norm unity and the latter has norm $\delta^3(0)$. So it is assumed that as the packet is broadened, its norm is steadily increased in such a way that we end up with a plane wave. In any case, the overall norm has no significance.

§ Actually f also depends on k; this dependence is not shown explicitly.

To get the differential cross section, we need the ratio of the probability flowing into $d\Omega$ per second to the incident current density. So what are $\mathbf{j}_{\rm sc}$ and $\mathbf{j}_{\rm inc}$, the incident and scattered current densities? Though we have repeatedly spoken of these quantities, they are not well defined unless we invoke further physical ideas. This is because there is only one current density $\mathbf{j}$ associated with $\psi_{\mathbf{k}}$ and it is *quadratic* in $\psi_{\mathbf{k}}$. So $\mathbf{j}$ is not just a sum of two pieces, one due to e^{ikz} and one due to $\psi_{\rm sc}$; there are cross terms.‡ We get around this problem as follows. We note that as $r\to\infty$, $\psi_{\rm sc}$ is negligible compared to e^{ikz} because of the $1/r$ factor. So we calculate the incident current due to e^{ikz} to be

$$\begin{aligned}|j_{\rm inc}| &= \left|\frac{\hbar}{2\mu i}(e^{-ikz}\nabla e^{ikz}-e^{ikz}\nabla e^{-ikz})\right| \\ &= \frac{\hbar k}{\mu}\end{aligned}\tag{19.2.13}$$

We cannot use this trick to calculate $\mathbf{j}_{\rm sc}$ into $d\Omega$ because $\psi_{\rm sc}$ never dominates over e^{ikz}. So we use another trick. We say that e^{ikz} is really an abstraction for a wave that is limited in the transverse direction by some $\rho_{\max}(\gg r_0)$. Thus in any realistic description, only $\psi_{\rm sc}$ will survive as $r\to\infty$ for $\theta\neq 0$.§ (For a given $\rho_{\max}$, the incident wave is present only for $\delta\theta\lesssim\rho_{\max}/r$. We can make $\delta\theta$ arbitrarily small by increasing the r at which the detector is located.) With this in mind we calculate (for $\theta\neq 0$)

$$\mathbf{j}_{\rm sc}=\frac{\hbar}{2\mu i}(\psi_{\rm sc}^*\nabla\psi_{\rm sc}-\psi_{\rm sc}\nabla\psi_{\rm sc}^*)\tag{19.2.14}$$

Now

$$\nabla=\mathbf{e}_r\frac{\partial}{\partial r}+\mathbf{e}_\theta\frac{\partial}{r\,\partial\theta}+\mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\tag{19.2.15}$$

The last two pieces in ∇ are irrelevant as $r\to\infty$. When the first acts on the asymptotic $\psi_{\rm sc}$,

$$\frac{\partial}{\partial r}f(\theta,\phi)\frac{e^{ikr}}{r}=f(\theta,\phi)ik\frac{e^{ikr}}{r}+O\left(\frac{1}{r^2}\right)$$

so that

$$j_{\rm sc}=\frac{\mathbf{e}_r}{r^2}|f|^2\frac{\hbar k}{\mu}\tag{19.2.16}$$

‡ We did not have to worry about this in one dimension because j due to $A\,e^{ikx}+B\,e^{-ikx}$ is $(\hbar k/\mu)(|A|^2-|B|^2)=j_{\rm inc}+j_{\rm ref}$ with no cross terms.

§ In fact, only in this more realistic picture is it sensible to say that the particles entering the detectors at $\theta\neq 0$ are scattered (and not unscattered incident) particles. At $\theta=0$, there is no way (operationally) to separate the incident and scattered particles. To compare theory with experiment, one extracts $f(\theta=0)$ by extrapolating $f(\theta)$ from $\theta\neq 0$.

Probability flows into $d\Omega$ at the rate

$$R(d\Omega)=\mathbf{j}_{sc}\cdot\mathbf{e}_r r^2 d\Omega$$
$$=|f|^2\frac{\hbar k}{\mu}d\Omega \qquad (19.2.17)$$

Since it arrives at the rate

$$j_{inc}=\hbar k/\mu \text{ sec}^{-1}\text{ area}^{-1}$$
$$\frac{d\sigma}{d\Omega}d\Omega=\frac{R(d\Omega)}{j_{inc}}=|f|^2 d\Omega$$

so that finally

$$\frac{d\sigma}{d\Omega}=|f(\theta,\phi)|^2 \qquad (19.2.18)$$

Thus, in the time-independent picture, the calculation of $d\sigma/d\Omega$ reduces to the calculation of $f(\theta, \phi)$.

After this general discussion, we turn to specific calculations. In the next section the calculation of $d\sigma/d\Omega$ is carried out in the time-dependent picture *to first order*. In Section 4, we calculate $d\sigma/d\Omega$ to first order in the time-independent picture. (The two results agree, of course.) In Section 5, we go beyond perturbation theory and discuss some general features of f for spherically symmetric potentials. Two-particle scattering is discussed in Section 6.

19.3. The Born Approximation (Time-Dependent Description)

Consider an initial wave packet that is so broad that it can be approximated by a plane wave $|\mathbf{p}_i\rangle$. Its fate after scattering is determined by the propagator $U(t_f\to\infty, t_i\to-\infty)$, that is, by the operator

$$S=\lim_{\substack{t_f\to\infty\\ t_i\to-\infty}} U(t_f, t_i)$$

which is called the *S matrix*. The probability of the particle entering the detector in the direction (θ, ϕ) with opening angle $d\Omega$ is the probability that the final momentum $\mathbf{p}_f$ lies in a cone of opening angle $d\Omega$ in the direction (θ, ϕ):

$$P(\mathbf{p}_i\to d\Omega)=\sum_{\mathbf{p}_f \text{ in } d\Omega}|\langle\mathbf{p}_f|S|\mathbf{p}_i\rangle|^2$$