The Born Approximation (Contd.)

Starting with the cross section formula

$$\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2$$

We try to estimate the scattering amplitude $f(\theta,\varphi)$ using TDPT result.

→ Assuming that scattering potential V is weak. (so that perturbation theory works).

→ Assuming ples. at $t \to \infty$ or $t \to -\infty$ are free particles with $\psi \sim e^{i\vec{k}\cdot\vec{r}} = e^{i\vec{p}\cdot\vec{r}/\hbar}$

Let, $$|\psi(t_i)\rangle \equiv |\vec{p}_i(t_i)\rangle = e^{-iHt_i}|\vec{p}_i\rangle$$

where $H \to$ some Hamiltonian

$$|\psi(t_f)\rangle \equiv |\vec{p}_f(t_f)\rangle = e^{-iHt_f}|\vec{p}_f\rangle$$

Then the transition probability that an incident particle gets scattered along solid $\angle^{le}$ $d\Omega$ with final momentum $\vec{p}_f$:

$$P(\vec{p}_i \to d\Omega) = \sum_{\substack{\text{all } \vec{p}_f \\ \text{in} \\ d\Omega}} |\langle \vec{p}_f | S | \vec{p}_i \rangle|^2$$

where,

$$\boxed{S \equiv \lim_{\substack{t_f \to \infty \\ t_i \to -\infty}} \underbrace{U(t_f, t_i)}} \leftarrow \boxed{S\text{-matrix}}$$

Unitary time-evolution operator $\Longleftrightarrow \sim e^{iH(t_f - t_i)}$ for some unknown H.

→ Recall from TDPT, we have derived the Fermi's golden rule for the rate of transition from infinite past ($t_i \to -\infty$) to infinite future ($t_f \to \infty$):

$$R_{i\to f} = \frac{P_{i\to f}}{T} = \frac{2\pi}{\hbar} |\langle f^0 | H^1 | i^0 \rangle|^2 \delta(E_f^0 - E_i^0 - \hbar\omega)$$

→ We must modify this eq. to apply it to our case:

$$R_{i\to f} \longrightarrow R_{i\to d\Omega}$$ (∵ we're interested in a given solid (i.e. ring $d\Omega$)

$$\therefore \quad P_{i\to f} \longrightarrow dP_{i\to d\Omega}$$

$$T \longrightarrow dt$$ (∵ Interaction happens only at scattering pt. ~ almost instantaneous.)

$$\delta(E_f - E_i - \hbar\omega) \longrightarrow \delta(E_f - E_i)$$ (∵ No external freq./periodic part.)

$$\langle f^0 | H^1 | i^0 \rangle \longrightarrow \langle \vec{p}_f | V | \vec{p}_i \rangle$$

$$\therefore \; R_{i\to d\Omega} = \frac{dP(\vec{p}_i \to d\Omega)}{dt} = \frac{2\pi}{\hbar} \underbrace{\sum_{\substack{\text{all } \vec{p}_f \\ \text{in } d\Omega}}}_{} |\langle \vec{p}_f | V | \vec{p}_i \rangle|^2 \delta\left(\frac{p_f^2}{2\mu} - \frac{p_i^2}{2\mu}\right)$$

Continuum limit in $\vec{p}$ space.

$$\int d^3\vec{p}_f = \int_0^\infty dp_f \cdot p_f^2 \oint d\Omega$$

$$\Rightarrow R_{i\to d\Omega} = \frac{2\pi}{\hbar} \int_0^\infty dp_f \, p_f^2 \, |\langle \vec{p}_f | V | \vec{p}_i \rangle|^2 \cancel{\delta(p_f^2} \, \delta(p_f^2 - p_i^2) \, d\Omega \cdot \underline{2\mu}$$

$$= \frac{2\pi\mu}{\hbar} \int_0^\infty d(p_f^2) \cdot p_f \, |\langle \vec{p}_f | V | \vec{p}_i \rangle|^2 \delta(p_f^2 - p_i^2) \, d\Omega$$

$$\Rightarrow \boxed{R_{i\to d\Omega} = \pm \frac{2\pi\mu}{\hbar} |\langle \vec{p}_f | V | \vec{p}_i \rangle|^2 p_i \, d\Omega}$$

→ Taking final & initial states of particles as free pl. state:

$$|\vec{p}_i\rangle \longrightarrow (2\pi\hbar)^{-3/2} e^{i\vec{p}_i \cdot \vec{r}/\hbar}$$

$$\Rightarrow \boxed{\langle \vec{p}_f | V | \vec{p}_i \rangle = \int d^3\vec{r} \frac{1}{(2\pi\hbar)^3} V(\vec{r}) \, e^{-i\vec{q}\cdot\vec{r}}}$$

$$\boxed{\vec{q} = \frac{1}{\hbar}(\vec{p}_f - \vec{p}_i)}$$

change in momentum due to scattering.

~~Note~~ Now,

$$\frac{d\sigma}{d\Omega} d\Omega \equiv \frac{R_{i\to d\Omega}}{|j_{inc}|}$$ → incident current

$$j_{inc} \equiv \frac{\hbar}{2\mu i}\left(\psi_i^* \nabla \psi_i - \psi_i \nabla \psi_i^*\right)$$

Here, $|j_{inc}| = \frac{1}{(2\pi\hbar)^3}\left|\left[e^{-i\vec{p}_i\cdot\vec{r}/\hbar}\vec{\nabla} e^{i\vec{p}_i\cdot\vec{r}/\hbar} - e^{i\vec{p}_i\cdot\vec{r}/\hbar}\vec{\nabla} e^{-i\vec{p}_i\cdot\vec{r}/\hbar}\right]\frac{\hbar}{2\mu i}\right|$

$$\Rightarrow |j_{inc}| = \frac{\hbar k_i}{\mu}\left(\frac{1}{2\pi\hbar}\right)^3 \sec^{-1}\text{area}^{-1} = \frac{p_i}{\mu}\left(\frac{1}{2\pi\hbar}\right)^3 \sec^{-1}\text{area}^{-1}$$

$$\therefore \frac{d\sigma}{d\Omega} d\Omega = \pm\frac{2\pi\mu}{\hbar}\frac{1}{(2\pi\hbar)^{6-3}}\left|\int d^3\vec{r}\, V(\vec{r})\, e^{-i\vec{q}\cdot\vec{r}}\right|^2 \cdot \frac{\cancel{p_i}\, d\Omega}{\cancel{p_i}\cdot\cancel{\left(\frac{1}{2\pi\hbar}\right)^3}}\mu$$

$$= \left|\frac{\mu}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}}V(\vec{r})\, d^3\vec{r}\right|^2 d\Omega$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \left|\frac{\mu}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}}V(\vec{r})\, d^3\vec{r}\right|^2} = |f(\theta,\varphi)|^2$$

Note: $|\vec{q}|^2 = |\vec{k}_f - \vec{k}_i|^2 = 2k^2(1-\cos\theta) = 4k^2\sin^2\theta/2$

Or $\boxed{|\vec{q}| \equiv q = 2k\sin(\theta/2)}$

$$\Rightarrow \boxed{f(\theta,\varphi) = f(\theta) = -\frac{\mu}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}'}V(\vec{r}')\, d^3\vec{r}'}$$

Born Approximation.
scattering amplitude.

In the case of spherically symmetric potential:

$$f(\theta,\varphi) = -\frac{\mu}{2\pi\hbar^2}\int e^{-iqr'\cos\theta'} V(r')\, d(\cos\theta')\, d\varphi'\, r'^2 dr'$$

$$= -\frac{2\mu}{\hbar^2}\int \frac{\sin qr'}{q} V(r')\, r'\, dr'$$

$$\boxed{= f(\theta)} \leftarrow \text{Since, } \boxed{q = |\vec{q}| = 2k\sin\theta/2}$$

Examples:

I : The Yukawa potential.

$$V(r) = \frac{g\, e^{-\mu_0 r}}{r}$$

$$\Rightarrow \quad rV(r) = g\, e^{-\mu_0 r}$$

$$\therefore\ f(\theta) = -\frac{2\mu g}{q\hbar^2}\int_0^\infty \frac{e^{iqr'} - e^{-iqr'}}{2i}\, e^{-\mu_0 r'}\, dr'$$

$$= \frac{-2\mu g}{\hbar^2(\mu_0^2 + q^2)}$$

$$\Rightarrow \quad \boxed{\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{4\mu^2 g^2}{\hbar^4\left[\mu_0^2 + 4k^2\sin^2\theta/2\right]^2}}$$

H/w: Set $g = Ze^2$ & $\mu_0 = 0$. What is $\frac{d\sigma}{d\Omega}$?