Partial Wave analysis

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→ So far, we have seen that:

$$\frac{d\sigma}{d\Omega} = |f(\theta,\varphi)|^2 = |f(\theta)|^2$$

$$= \left| -\frac{2\mu}{\hbar^2} \int \frac{\sin qr'}{q} V(r') r' dr' \right|^2$$

where $q = 2k\sin\theta/2$

→ Actually,

$$f(\theta) = f(\theta, \underline{\underline{k}}) \;!! \quad \left[\because\; q = 2k\sin\theta/2\right]$$

→ Practically, a collider has beam of particles in a given momentum range.

⇒ It is often the "best practice" in collider physics to work with k-dependent terms directly instead of $f(\theta,k)\ \forall k$.

→ Consider the expansion of $f(\theta,k)$ in terms of the Legendre polynomials:

$$f(\theta,k) = \sum_{l=0}^{\infty} (2l+1)\, \underbrace{a_l(k)}_{l\text{th partial wave amplitude.}}\, \overbrace{P_l(\cos\theta)}^{\text{Legendre polynomials.}}$$

→ In this representation, we have seperated the k & θ dependence.

→ $a_l(k)$ is a measure of the scattering in the angular momentum "l" sector.

$$\underbrace{f(\theta, k)}_{1-\text{fn.}} \longrightarrow \underbrace{a_l(k),\ P_l(\cos\theta)}_{\infty\ \#\text{ of fns.}};\ l=0,1,\ldots\infty.$$

→ But, $a_l(k)$ helps ~~to~~ segregate the scattering wrt the momentum band k of plen-

→ Incident momentum → $\hbar l \sim \hbar k \underbrace{\rho}_{\text{Impact param.}}$

→ The range of scattering (in momentum!):

$$l_{max} \sim k\rho_{max} \sim k\underbrace{r_0}_{\text{range of potential.}}$$

→ After scattering,

$$\psi_{sc} \sim e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta)$$

$$j_l(kr) \xrightarrow{r\to\infty} \frac{\sin(kr - l\pi/2)}{kr}$$

$$\Rightarrow e^{ikz} \xrightarrow[r\to\infty]{} \frac{1}{2ik}\sum_{l=0}^{\infty} \underbrace{i^l}_{e^{il\pi/2}}(2l+1)\left(\frac{e^{i(kr-l\pi/2)}}{r} - \frac{e^{-i(kr-l\pi/2)}}{r}\right)P_l\cos\theta$$

Free particle.

$$= \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left(\underbrace{\frac{e^{ikr}}{r}}_{\sim\text{incoming wave packet}} - \underbrace{\frac{e^{-i(kr-l\pi)}}{r}}_{\sim\text{outgoing wavepacket}}\right)P_l(\cos\theta)$$

$\boxed{\parallel \text{ phase diff} \sim l\pi!}$

$\rightarrow$ Taking into account the effect of scattering:

$\rightarrow$ We expect the scattered ptle to again have $\sim$ plane wave form, but with some phase shift δ.

$$\Rightarrow \underbrace{R_l(r)}_{\text{radial part of }\psi} = \frac{U_l(r)}{r} \xrightarrow[r\to\infty]{} \frac{A_l \sin[kr - l\pi/2 + \delta_l(k)]}{r}$$

$\therefore$ ∥ar to free ptle case:

$$\Rightarrow \psi_{\vec{k}}(\vec{r}) \xrightarrow[r\to\infty]{} \sum_{l=0}^{\infty} A_l \frac{\left(e^{i(kr-l\pi/2+\delta_l)} - e^{-i(kr-l\pi/2+\delta_l)}\right)}{r} P_l(\cos\theta)$$

$$\downarrow$$

$\sim e^{ikz}\ (r\to\infty)$
[plane wave]

$$\Rightarrow \quad A_\ell = \frac{2\ell+1}{2ik}\, e^{i(\ell\pi/2 + \delta_\ell)}$$

$$\Rightarrow \quad \psi_{\vec{k}}(\vec{r}) \xrightarrow[r\to\infty]{} \frac{1}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)\left[e^{ikr}e^{2i\delta_\ell} - e^{-i(kr-\pi\ell)}\right]P_\ell(\cos\theta)$$

(adding):

$$+\frac{e^{ikr}}{r} - \frac{e^{ikr}}{r}$$

$$\Rightarrow \quad \psi_k(\vec{r}) \xrightarrow[r\to\infty]{} \underbrace{e^{ikz}}_{\psi_{free}\sim\psi_{inc}} + \underbrace{\left[\sum_{\ell=0}^{\infty}(2\ell+1)\left(\frac{e^{i2\delta_\ell}-1}{2ik}\right)P_\ell(\cos\theta)\right]\frac{e^{ikr}}{r}}_{\psi_{sc}.}$$

$\Rightarrow$ Comparing :

$$\boxed{a_\ell(k) = \frac{e^{2i\delta_\ell}-1}{2ik}}$$

$$\therefore\ f(\theta,\varphi) \longrightarrow \underbrace{a_\ell(k)}_{\frac{e^{2i\delta_\ell}-1}{2ik}}\ ;\ \underbrace{P_\ell(\cos\theta)}_{\text{known.}}$$

$\delta_\ell \rightarrow$ only unknown.

<u>We define:</u> $\boxed{S_\ell(k) \equiv e^{2i\delta_\ell(k)}}$ $\leftarrow$ partial wave S-matrix.

Re-writing:

$$a_\ell(k) = \frac{e^{2i\delta_\ell}-1}{2ik} = \frac{e^{i\delta_\ell}\sin(\delta_\ell)}{k}$$

$$\Rightarrow \boxed{f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin(\delta_\ell)\,P_\ell(\cos\theta)}$$

Partial wave expansion of the scattering amplitude.

→ An interesting result:

Using the known result for ~~cros~~ total cross-section,

$$\sigma = \int |f|^2 d\Omega = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$

We note: at $\theta=0 \Rightarrow P_\ell(\cos 0) = 1$

$$f(0) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\left[\cos(\delta_\ell)\sin(\delta_\ell)+i\sin^2(\delta_\ell)\right]$$

$$\Rightarrow \operatorname{Im}[f(0)] = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2(\delta_\ell)$$

$$\Rightarrow \frac{4\pi}{k}\operatorname{Im}[f(0)] = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2(\delta_\ell)$$

$$\Rightarrow \boxed{\sigma = \frac{4\pi}{k}\operatorname{Im}[f(0)]}$$

← The total cross-section!

The optical theorem.