Department of Physics, IIT Patna

PH623 - Introduction to general relativity and cosmology
(Instructor: Sandeep Aashish)

Spring 2024 February 5, 2024

Assignment 2

**Due on Mon 12/2/2024.*

1. How does gravitational field affect tides on Earth's oceans? Read about this from various sources restricting to general relativistic connection, and summarise in a short paragraph. Why a uniform external gravitational field would **not** raise tides on Earth?

2. Consider the $\mathcal{R}^3$ manifold (3D space) with the flat Euclidean metric, and two coordinate systems namely the cartesian and the spherical polar.
   (a) What is the respective metric for each of the coordinate systems?
   (b) Consider a particle moving along a parametrized curve given by $x(\lambda) = \cos\lambda$, $y(\lambda) = \sin\lambda$, and $z(\lambda) = \lambda$. Express the path of the curve in the (r, θ, ϕ) coordinates.
   (c) Calculate the component of the tangent vector to the curve in both the cartesian and spherical polar coordinate system.

3. Assuming a 2D space, with two coordinate systems: cartesian (x, y) and polar (r, θ), calculate all elements of the transformation matrices $\Lambda^{\alpha'}_{\beta}$ and $\Lambda^{\mu}_{\nu'}$ for the transformation from cartesian (unprimed indices) to polar (primed indices).

4. A π meson has an average lifetime of 26ns in its own frame of reference. If it moves with the speed 0.95c with respect to Earth, what is the lifetime as measured by an observer at rest on Earth? What is the average distance it would travel as measured by an observer at rest on Earth?

5. Derive the formula for Lorentz velocity transformation: $u'_x = (u_x - v)/(1 - u_x v/c^2)$, where u_x is the velocity of an object measured in frame S and v is the velocity of frame S' wrt S. Assume that all velocities are along x-axis.

6. If V_ν is an arbitrary covariant vector and $T^{\mu\nu}V_\nu$ is known to be a contravariant **vector** for some $T^{\mu\nu}$, then show that:

$$\left(T'^{\mu\nu} - T^{\lambda\sigma} \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} \right) V'_\nu = 0.$$

   Based on the above result, can you say that $T_{\mu\nu}$ is a tensor? Why?

7. Is $\Gamma^{\alpha}_{\mu\nu}$ a tensor? Use the transformation properties of the metric $(g_{\mu\nu})$ and the general coordinate transformations to show how $\Gamma^{\alpha'}_{\mu'\nu'}$ transforms.

8. How does the covariant derivative $\nabla_\mu V^\nu$ transform under general coordinate transformation?

9. From the definition of the Christoffel connections in terms of the metric, show that: ${\Gamma^\mu}_{\mu\lambda} = \frac{1}{\sqrt{|g|}}\partial_\lambda \sqrt{|g|}$, where $|g| = \det(g_{\mu\nu})$. This implies: $V^\mu_{;\mu} = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}V^\mu)$

10. For a metric given by the invariant length element: $ds^2 = dr^2 + r^2 d\theta^2$, Find out the components of $\Gamma^\alpha_{\mu\nu}$.

11. Naively, one would construct a 4-dimensional volume element as follows: $dV = d^4x = dx^0 dx^1 dx^2 dx^3$. How does dV transform under GCT? How can you make it invariant?