# Redshift

→ Galaxies observed in the visible spectrum are red-shifted.

$$z \equiv \frac{\lambda_{ob} - \lambda_{em}}{\lambda_{em}} > 0 \leftarrow \text{redshift}$$

→ Hubble's law:

$$\boxed{z = \frac{H_0}{c}\, r} \rightarrow \text{distance from earth.}$$

$H_0 \rightarrow$ Hubble constant.

→ From observations,

$$H_0 = 68 \pm 2 \text{ km s}^{-1}\text{ Mpc}^{-1}$$

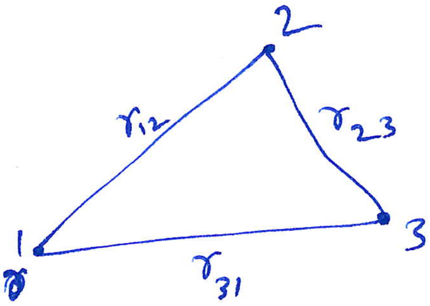
⇒ The farther galaxies are moving farther away from us!!

But this doesn't mean we are the center of the universe.

→ Homogeneity implies that an observer in the distant galaxy would observe the same.

→ Consider three galaxies at $\vec{r}_1, \vec{r}_2, \vec{r}_3$:

$$r_{12} \equiv |\vec{r}_1 - \vec{r}_2|$$
$$r_{23} \equiv |\vec{r}_2 - \vec{r}_3|$$
$$r_{31} \equiv |\vec{r}_3 - \vec{r}_1|$$




→ For a homogeneous & isotropic expansion, all these galaxies must move uniformly i.e

$$r_{12}(t) = a(t)\, r_{12}(t_0)$$
$$r_{23}(t) = a(t)\, r_{23}(t_0)$$
$$r_{31}(t) = \underbrace{a(t)}_{\boxed{\text{scale factor}}}\, r_{31}(t_0)$$

→ We define the scale factor s.t. $\underbrace{a(t_0) = 1}_{\text{present time.}}$

⇒ W.r.t. the observer in galaxy 1:

$$v_{12}(t) = \frac{d r_{12}}{dt} = \dot{a}\, r_{12}(t_0) = \frac{\dot{a}}{a} \cdot a\, r_{12}(t_0)$$
$$\equiv H(t)\, \cancel{a\, r_{12}}\, r_{12}(t)$$

$$\Rightarrow \boxed{v_{12}(t) = H(t)\, r_{12}(t)}$$

where $H(t) \to$ Hubble parameter ~ Hubble constant at time $t$.

$\therefore$ At present time $t = t_0$ :

$$v(t_0) = H(t_0)\, r(t_0)$$

$$\Rightarrow \boxed{v = H_0 r}$$

→ Assuming that this v is constant (i.e in absence of external forces),

$$t_0 = \frac{r}{v} = \frac{r}{H_0 r} = \underbrace{\frac{1}{H_0}}_{\downarrow} \leftarrow \text{Hubble time.}$$

Time since the galaxies were in causal contact.

* $t_0 \simeq 14.38 \pm 0.42$ Gyr. ~ 14 billion yrs.

$\Downarrow$

Big bang model of the universe.

→ ~~The current "standard" model of universe -~~

~~ΛCDM →~~

[Q] Using our knowledge of general relativity, how to model such a homogeneous, isotropic ← expanding universe ?

→ Friedmann - Lemaitre - Robertson - Walker metric (FLRW/ FRW metric)

Recall: $g_{\mu\nu}(x) \leftarrow$ describes the geometry of spacetime.

Considering that the universe at large length-scales $\Longleftrightarrow$ the ~~~~ early universe is

(1) homogeneous } $\to$ diagonal elements.

(2) Isotropic }

(3) Expanding $\longrightarrow$ scale factor $a(t)$

(4) May admit curvature. $\longrightarrow S_k(r)$

$\to$ FLRW metric:

$$\boxed{ds^2 = -c^2dt^2 + (a(t))^2\left[dr^2 + S_k(r)^2 d\Omega^2\right]}$$

The metric describing the early universe!

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\varphi^2$

$$S_k(r) = \begin{cases} R\sin(r/R), & k=+1 \\ r, & k=0 \\ R\sinh(r/R), & k=-1 \end{cases}$$

→ In the natural units, $(c=1)$ :

$$ds^2 = -dt^2 + a(t)^2\left[dr^2 + S_k^2(r)\, d\Omega^2\right]$$

$$= -a(t)^2 \cdot \left(\frac{dt}{a(t)}\right)^2 + a(t)^2\left[\cdots\right]$$

$$\equiv \underbrace{a^2}_{\text{co-moving scale factor!}} \underbrace{\left[-d\eta^2 + dr^2 + S_k^2\, d\Omega^2\right]}_{\sim \text{ Minkowski}}$$

→ Here,

$$d\eta \equiv \frac{dt}{a(t)} \leftarrow ??$$

Consider the null geodesic, i.e. the trajectory of light, assuming light travels along radial dir:

$$\Rightarrow \quad ds^2 = 0 = -dt^2 + a^2 dr^2$$

$$\Rightarrow \quad dr^2 = \left(\frac{dt}{a}\right)^2$$

$$\Rightarrow \quad |dr| = \left|\frac{dt}{a}\right|$$

$$\Rightarrow \quad |dr| = \int \frac{dt}{a} \leftarrow \text{distance travelled by light in time "dt".}$$

$$= \eta$$

$$\Rightarrow \eta \equiv \int_0^t \frac{dt}{a(t)} \leftarrow \text{Comoving distance.}$$

⇕ Conformal time.

# The FRW metric

$$ds^2 = a^2(t)\left[-c^2\underbrace{\left(\frac{dt}{a}\right)^2}_{\equiv d\eta} + dr^2 + S_k(r)^2 d\Omega^2\right]$$

$\eta \rightarrow$ conformal time.

$(r, \theta, \varphi) \rightarrow$ comoving coordinates. $\leftarrow$ Independent of scale factor!

$\Downarrow$

Comoving distance doesn't increase with time.

Excludes the effect of expanding universe..

$$k = \begin{cases} +1, & \text{+ve curvature} \\ 0, & \text{flat} \\ -1, & \text{-ve curvature.} \end{cases}$$

## → Proper distance:

At a fixed time $t$:

$$ds^2 = a^2(t)\left[dr^2 + S_k(r)^2 d\Omega^2\right]$$

Along the geodesic:

$$ds^2 = a^2(t)\, dr^2$$

∴ We define, the proper distance $d_p$:

$$d_p(t) = a(t)\int_0^r dr = a(t)\, r$$

$$\Rightarrow \quad \dot{d}_p = \frac{d}{dt}(a(t)r)$$

But $r \neq r(t)$,

$$\Rightarrow \quad \dot{d}_p = \dot{a}(t)\, r = \frac{\dot{a}}{a} d_p$$

$$\Rightarrow \quad \boxed{\dot{d}_p = H d_p} \equiv v_p$$

velocity of a receding galaxy.

→ What if $v_p = c$?

$$\Rightarrow \quad c = H d_p$$

$$\Rightarrow \quad \boxed{d_p = \frac{c}{H}}$$

Hubble radius.

The imaginary boundary within which ples move w/ speed $v < c$!!

doesn't mean that ples beyond $r_H$ are unobservable!!

At current time,

$$\boxed{d_p \equiv r_H = \frac{c}{H_0} \approx 4380 \pm 130 \text{ Mpc}.}$$

Now, Recall: we started with the question of how to understand the dynamics of the universe at large scales (early times)?

→ We know GR governs gravity at large scale

→ We have already constructed FRW metric — describing early universe.

→ The motivation we discussed at the start of this course!!

→ So, if FRW metric describes the geometry of early universe, what does the dynamics look like?

→ The Einstein equations!!

Recall: $S = \int d^4x \sqrt{-g}\left[R + \mathcal{L}_m\right]$

& $\frac{\delta S}{\delta g^{\mu\nu}} = 0$

$\Rightarrow \quad \boxed{G_{\mu\nu} = 8\pi G\, T_{\mu\nu}} \quad (c=1)$

$T_{\mu\nu}$ → from $\mathcal{L}_m$

$G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} R$
(gravity part)

→ Since the metric $g_{\mu\nu}$ has 10 independent components,

$G_{\mu\nu} - 8\pi G\, T_{\mu\nu} = 0 \longrightarrow$ 10 independent non-linear equations!
& $2^{nd}$ order diff. eq.

→ Fortunately, at least at early times there are symmetries in the universe we can exploit to simplify the Einstein's equation.

→ $T_{\mu\nu}$: stress-energy tensor — we assume the early universe (at large scales) is filled with isotropic & homogeneous perfect gas.

↳ modelling the primary matter content of universe.

$\Rightarrow$ The measurable quantities for such an ideal gas are (1) pressure $p(t)$ and (2) energy density $\rho(t)$ $\Big\}$ $\leftarrow$ only a fn. of time!

$\rightarrow$ No bulk velocity of gas, since it would break isotropy.

Ily on the gravity side of Einstein eq., we already have

$$ds^2 = -\cancel{c^2}^{\,1} dt^2 + a^2(t)\left[dr^2 + S_k^2(r)\, d\Omega^2\right]$$

$$S_k(r) = \begin{cases} R_0 \sin(r/R_0), & k=+1 \\ r, & k=0 \\ R_0 \sinh(r/R_0), & k=-1 \end{cases}$$

$\Rightarrow$ Only relevant parameters are $a(t)$, $k$ & $R_0$.

$\rightarrow$ Therefore, our goal (and that of early universe cosmology) is to find out how $a(t)$, $k$ & $R_0$ are related to $\rho(t)$ & $P(t)$?

$$\therefore \quad T^\mu{}_\nu = \begin{bmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix}$$

## Calculating $G_{\mu\nu}$:

→ Let's first look at 00 component:

$$R_{00} - \frac{1}{2} g_{00} R = 8\pi G\, T_{00}$$

$$R_{00} = \cancelto{0}{\Gamma^{\alpha}_{00,\alpha}} - \Gamma^{\alpha}_{\alpha 0,0} + \Gamma^{\alpha}_{\beta\alpha}\cancelto{0}{\Gamma^{\beta}_{00}} - \Gamma^{\alpha}_{\beta 0}\Gamma^{\beta}_{0\alpha}$$

$$= \cancel{0} - \Gamma^{i}_{0i,0} - \Gamma^{i}_{j0}\Gamma^{j}_{0i}$$

Here,

$$\Gamma^{0}_{00} = 0 = \Gamma^{0}_{0i} = \Gamma^{0}_{i0}$$

$$\Gamma^{0}_{ij} = \delta_{ij}\dot{a}a$$

$$\Gamma^{i}_{0j} = \Gamma^{i}_{j0} = \delta_{ij}\frac{\dot{a}}{a}$$

$$\Rightarrow \quad R_{00} = -\delta_{ii}\frac{\partial}{\partial t}\left(\frac{\dot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right)^2 \delta_{ij}\delta_{ij}$$

$$= -3\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right] - 3\left(\frac{\dot{a}}{a}\right)^2$$

$$= -3\frac{\ddot{a}}{a}$$

H/W $\quad \boxed{R_{ij} = ?} \quad \left( = \delta_{ij}\left(2\dot{a}^2 + a\ddot{a}\right)\right)$

$$\therefore \quad R = g^{\mu\nu} R_{\mu\nu}$$

$$= -R_{00} + \frac{1}{a^2} \cancel{R_{ij}} \, R_{ii}$$

$$\Rightarrow \boxed{R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]}$$

Finally, for the 00 component of E. eq:

$$R_{00} - \frac{1}{2} g_{00} R = 8\pi G T_{00}$$

$$\Rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho}$$

First Friedmann equation!