The Friedmann Equations

Recall: The first Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \leftarrow \text{True for } k=0 \text{ (Flat)}$$

$\boxed{\text{H/W}}$ How would this eq. change if $k \neq 0$?

$\Gamma^i_{jk} = ?$ . . . .

→ For general curvature of universe,

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{R_0^2}\frac{1}{a^2}}$$

where, $R_0 \rightarrow$ radius of curvature.

$k \rightarrow$ Curv. param. $(\pm 1, 0)$

→ If the universe is spatially flat $(k=0)$:

$$\Rightarrow H^2 = \frac{8\pi G}{3}\rho$$

$$\Rightarrow \boxed{\rho = \frac{3}{8\pi G}H^2 \equiv \varepsilon_c} \leftarrow \text{Critical density.}$$

→ If, the observed energy density $\rho(t) > \varepsilon_c(t)$,

$\Rightarrow k = +1$ i.e positively curved universe.

Illy $\rho(t) < \varepsilon_c(t) \Rightarrow k = -1$ i.e negatively curved.

The current value of critical density (in SI units):

$$\varepsilon_{c,0} = \frac{3c^2}{8\pi G} H_0^2 \simeq 4870 \pm 290 \text{ MeV m}^{-3}$$

$\sim$ $\boxed{1 \text{ p}^+ \text{ per } 200 \text{ litres}}$ ← Very low density!

A 200 litre drum of water has $\sim 10^{29}$ pts!!

∴ Our universe at large scales is <u>almost flat</u>.

→ We define a <u>density parameter</u>:

$$\boxed{\Omega(t) \equiv \frac{\rho(t)}{\varepsilon_c(t)}}$$

From observations, $\boxed{\Omega_0 \simeq 1 \pm 0.005}$

∴ In terms of Ω, the first Friedmann eq:

$$H^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{R_0^2 a^2}$$

$$= H^2 \cdot \frac{\rho(t)}{\varepsilon_c(t)} - \frac{k}{R_0^2 a^2}$$

$$\Rightarrow \boxed{1 - \Omega = -\frac{k}{R_0^2 a^2 H^2}}$$

<u>NOTE</u>: → If $\Omega < 1$ at any time t, then $\Omega < 1 \ \forall t$ ∵ RHS sign is fixed (k)!

Ily $\Omega > 1$ at $t_0 \Rightarrow \Omega > 1 \ \forall t$.

The evolution of $a(t)$: Second Friedmann eq.

→ Before proceeding, we derive ~~another~~ a key relation b/w ρ & P (pressure) :

→ ~~Consider~~ The Friedmann equation has two unknowns : $a(t)$ & $\rho(t)$

→ To solve any dynamical eq. of $a(t)$, one needs the initial condition $a(t_0)$ & $\dot{a}(t_0)$ as well as $\rho(t_0)$.

→ So, ideally we require atleast 3 eq.: '2' constraint eq. + 1 2nd order dynamical eq.

→ First Friedmann eq. is a constraint eq.

Ily we now consider a version of energy conservation in the universe

(* Cosmology involves multiple disciplines of physics!)

→ From thermodynamics (first law) :

$$dQ = dE + P dV$$

For a homogeneous universe, there's no bulk heat flow ~ Adiabatic universe :

$$\Rightarrow \quad dQ = 0$$

$$\Rightarrow \quad \boxed{\frac{dE}{dt} + P\frac{dV}{dt} = 0}$$

→ Considering the universe as an expanding sphere of comoving radius r_s,

the proper radius, $\boxed{R_s = a(t)\, r_s}$

$$\Rightarrow \quad V(t) = \frac{4\pi}{3} R_s^3 = \frac{4\pi}{3} a^3 r_s^3$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{4\pi}{3} \cdot 3 a^2 \dot{a}\, r_s^3$$

$$\Rightarrow \quad \boxed{\dot{V} = \left(\frac{4\pi}{3} a^3 r_s^3\right) 3 \frac{\dot{a}}{a} = 3HV}$$

→ If the energy density is $\rho(t)$, then the internal energy of sphere:

$$E(t) = V(t)\, \rho(t)$$

$$\Rightarrow \quad \dot{E} = \dot{V}\rho + V\dot{\rho} = 3H\rho V + V\dot{\rho}$$

$$\Rightarrow \quad \boxed{\dot{E} = V(\dot{\rho} + 3H\rho)}$$

→ Substituting in the 1st law eq:

$$\Rightarrow \quad V(\dot{\rho} + 3H\rho) + P(3HV) = 0$$

$$\Rightarrow \quad \boxed{\dot{\rho} + 3H(\rho + P) = 0} \leftarrow \boxed{\text{Fluid equation.}}$$

The evolution of $a(t)$:

From the first Friedmann eq:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{\kappa}{R_0^2}$$

$$\Rightarrow \quad 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\left(\dot{\rho}a^2 + 2a\dot{a}\rho\right)$$

$$\Rightarrow \quad \frac{\ddot{a}}{a} = \frac{8\pi G}{3}\left[\frac{\dot{\rho}a}{2\dot{a}} + \rho\right]$$

Using the fluid equation: $\dot{\rho} = -3H(\rho + P)$

$$\Rightarrow \quad \frac{\ddot{a}}{a} = \frac{8\pi G}{3}\left[\frac{-1}{2\cancel{H}}\cdot 3\cancel{H}(\rho+P) + \rho\right]$$

$$\Rightarrow \quad \boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}$$

Acceleration eq. OR 2nd Friedmann eq.

H/W Derive the 2nd Friedmann eq. directly from the ~~Ein.~~ ij component of Einstein eq.

Q → How does the expansion change over time?

→ What happens when $P < -\frac{1}{3}\rho$ & $\rho > 0$?

Summary:

(1) The (first) Friedmann eq:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{R_0^2 a^2}$$

(2) The fluid equation:

$$\dot{\rho} + 3H(\rho + P) = 0$$

(3) The Acceleration eq:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

Q How many independent equations do we have?

→ 2 independent eq., 3 unknowns: $\boxed{\rho, P, a}$

⇒ One needs another constraint equation, the equation of state

$$\sim \quad P \overset{?}{=} P(\rho) \quad ?$$

→ In cosmology, it suffices to use

$$\boxed{P = w\rho}$$

where $w \rightarrow$ dimensionless parameter.

Figuring out w:

→ Consider, a low-density ~~gas~~ non-relativistic gas of massive particles.

Perfect gas law: $PV = N k_B T$

$$\Rightarrow P = \underbrace{\left(\frac{N}{V}\right)}_{\text{number density}} k_B T$$

$$= \frac{N\mu}{\mu V} k_B T \quad \mu \rightarrow \text{mean mass of gas ptcls.}$$

$$\equiv \frac{\sigma}{\mu} k_B T \quad \sigma \rightarrow \text{mass density}$$

→ Using the energy mass relation for non-rel. ptcls:

$$\rho \approx \sigma c^2 \approx \sigma \text{ (in cosm unit)}$$

$$\Rightarrow \boxed{P = \frac{k_B T}{\mu} \rho}$$

→ At temperature T, the rms vel. is given by:

$$3 k_B T = \mu \langle v^2 \rangle$$

$$\Rightarrow \boxed{P = \frac{\langle v^2 \rangle}{3} \rho}$$

$$\therefore P_{NR} = \underbrace{\left(\frac{\langle v^2 \rangle}{3}\right)}_{w} \rho_{NR} \leftarrow \text{Non-relativistic.}$$

$\because \langle v^2 \rangle_{NR} << 1$ for NR $\Rightarrow \boxed{P_{NR} = w \rho_{NR}}$, $\boxed{w << 1}$

Illy for a relativistic gas, say, of photons.

$\rho = p c$ → momentum.

$\Rightarrow \quad P_{rel.} = w \rho_{rel.}$

where $\boxed{w = \frac{\langle v^2 \rangle}{3} \approx \frac{1}{3}}$

$\Rightarrow \quad \boxed{P_{rel} = \frac{1}{3} \rho_{rel}}$ ← relativistic gas.

∴ The parameter w then describes the composition of universe.

→ $w \approx \frac{1}{3} \Rightarrow$ photon-filled universe.

→ $w = 0 \Rightarrow$ Completely static matter (N.R.) filled universe. (cold matter)

$\Rightarrow \quad w \approx \frac{1}{3}$ ← "Radiation dominated" universe.

$w \approx 0$ ← "Matter dominated" universe.

→ Now, consider the case when $P < -\frac{1}{3} \rho$:

$\Rightarrow \quad \boxed{w < -\frac{1}{3}}$ ← What is the composition of this universe ??

Dark energy !!

Would indicate cosmological constant.