The Cosmic Dynamics

We have derived 3 fundamental eqs. of cosmology:

(1) $$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R_0^2 a^2}$$

(2) $$\dot{\rho} = -3H(\rho + P)$$

(3) $$P = w\rho$$

along with the acceleration eq. (derived fr 1 & 2):

(4) $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

$\boxed{Q}$ How to re-write (4) in terms of $H(t)$? $\sqrt{H/W}$

→ Now, let's solve these equations for various cases of EoS:

↓

Evolution of energy density:

→ Possible (likely) composition of the universe:

~~m~~ non-relativistic matter (w=0), ~~r~~

radiation ($w = \frac{1}{3}$)

dark energy ($w = -1$), - - -

$\Rightarrow$ Total energy density, $\rho = \sum_i \rho_i$ → for each component i.

$\Rightarrow$ $P = \sum_i w_i \rho_i = \sum_i P_i$

∴ The fluid equation:

$$\dot{\rho} + 3H(\rho + P) = 0$$

$$\Rightarrow \sum_i \left[\dot{\rho}_i + 3H(\rho_i + P_i)\right] = 0$$

$$\Rightarrow \boxed{\dot{\rho}_i + 3H(\rho_i + P_i) = 0}$$ ← Fluid eq. holds for each component

Now, applying EoS:

$$P_i = w_i \rho_i$$

$$\Rightarrow \dot{\rho}_i + 3H(\rho_i + w_i \rho_i) = 0$$

$$\Rightarrow \frac{d\rho_i}{\rho_i} = -3H(1+w_i)\,dt$$

$$= -3\frac{da}{a\cancel{dt}}(1+w_i)\,\cancel{dt}$$

$$\Rightarrow \frac{d\rho_i}{\rho_i} = -3(1+w_i)\frac{da}{a}$$

$$\Rightarrow \ln \rho_i = -3(1+w_i)\ln a + C$$

$$\Rightarrow \boxed{\rho_i(a) = \rho_{i,0}\, a^{-3(1+w_i)}}$$ ← Assuming w_i is a const.

Case I: Relativistic matter / radiation: $w_i = \frac{1}{3}$

$$\Rightarrow \boxed{\rho_{rad}(a) = \rho_{rad,0}\, a^{-4}}$$

Case II: Non-relativistic matter: $w_i = 0$

$$\Rightarrow \boxed{\rho_m(a) = \rho_{m,0}\, a^{-3}}$$

H/W: Read Sec 5.1 from Ryden. (Eq. 5.12 ... 5.20)

→ Solving the Friedmann eq:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \quad \leftarrow \text{for flat universe case. } (k=0)$$

$$\Rightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_{i,0}\, a^{-3(1+w_i)}$$

$$\Rightarrow \dot{a}^2 = \frac{8\pi G}{3}\rho_{i,0}\, a^{-1-3w_i}$$

$$\Rightarrow \frac{da}{dt} = \sqrt{\frac{8\pi G}{3}\rho_{i,0}}\; a^{-(1+3w_i)/2}$$

$$\Rightarrow \int da \cdot a^{(1+3w_i)/2} = \int \sqrt{\frac{8\pi G\rho_{i,0}}{3}}\, dt$$

$$\Rightarrow \frac{2a^{3(1+w_i)/2}}{3(1+w_i)} = \sqrt{\frac{8\pi G\rho_{i,0}}{3}}\, t + C$$

$$\Rightarrow a(t) = \cdots \quad \leftarrow \text{Some complicated fn. of time.}$$

→ Therefore, we don't always explicitly solve for $a(t)$, instead $a(t) \sim$ measurement of time.

<u>Example</u>: In a multiple component universe,

$$\dot{a}^2 = \frac{8\pi G}{3}\sum_i \rho_{i,0}\, a^{-1-3w_i} - \frac{k}{R_0^2}$$

$\uparrow$

<u>Very complicated</u>.

<u>Class exercise</u>: Solve the dynamical equations for an empty universe. [Read Sec. 5.2, Ryden]

Empty universe $\rightarrow$ No matter, radiation, Λ ....

$$\Rightarrow \boxed{\rho = 0}$$

$$\Rightarrow \frac{\dot{a}^2}{a^2} = -\frac{k}{R_0^2 a^2}$$

$$\Rightarrow \boxed{\dot{a}^2 = -\frac{k}{R_0^2}}$$

$\rightarrow$ For real $a(t)$, $\boxed{k = -1}$

$$\Rightarrow \boxed{\dot{a}(t) = 0, \quad k = 0} \leftarrow \boxed{\text{Flat, static universe.}}$$

<u>OR</u>

$$\boxed{\dot{a}(t) \neq 0, \quad k = -1} \leftarrow \boxed{\text{Negatively curved universe.}}$$

Taking $k = -1$:

$$\Rightarrow \boxed{\dot{a}(t) = \pm \frac{1}{R_0}} \leftarrow \boxed{\text{Expanding or contracting}}$$

$$\Rightarrow \boxed{a(t) = \frac{t}{t_0}}, \quad t_0 \equiv R_0$$

$a(t)$ is closely related to time!

→ Let's get back to solving the Friedmann eq:

Considering a single-component universe.

$$\therefore \frac{2}{3(1+w)} a^{3(1+w)/2} = \sqrt{\frac{8\pi G \rho_0}{3}}\, t + C$$

$$\Rightarrow \boxed{a(t) = \left(\frac{t}{t_0}\right)^{2/3(1+w)}} \text{ with appropriate } t_0.$$

→ NOTE: Here, $w \neq -1$ (dark energy)

$$\boxed{t_0 = \frac{1}{1+w}\left(\frac{1}{6\pi G \rho_0}\right)^{1/2}} \leftarrow \text{Current time corresponding to } a_0 = 1$$

$\Downarrow$

[The age of universe]

$$\Rightarrow H_0 \equiv H(t_0) = \frac{\dot{a}(t_0)}{a(t_0)} = \frac{2}{3(1+w_0)} t_0^{-1}$$ ← Hubble const. at present time.

$$\Rightarrow \boxed{t_0 = \frac{2}{3(1+w)} H_0^{-1}}$$ ← Age of the universe in terms of H_0.

$$\equiv \frac{2}{3(1+w)} t_H \; ,$$ where, $t_H \rightarrow$ Hubble time.

∴ → Case: $w > -\frac{1}{3} \Rightarrow t_0 < t_H$ → universe is <u>younger</u> than Hubble time.

→ $w < -\frac{1}{3} \Rightarrow t_0 > t_H$ → universe is <u>older</u> than Hubble time.

→ Empty universe $\Rightarrow t_0 = t_H$

<u>Now, from the redshift calculation:</u>

$$1+z = \frac{a(t_0)}{a(t_e)},$$ $t_0 \rightarrow$ current time.
$t_e \rightarrow$ time of emission of light.

$$= \left(\frac{t_0}{t_e}\right)^{2/(3+3w)}$$

$$\Rightarrow t_e = \cancel{\frac{2}{3(1}} \frac{t_0}{(1+z)^{3(1+w)/2}} = \frac{2}{3(1+w) H_0} \frac{1}{(1+z)^{3(1+w)/2}}$$

∴ The proper distance of galaxy:

$$d_p(t_0) = \int_{t_e}^{t_0} \frac{dt}{a(t)} = t_0 \frac{3(1+w)}{1+3w} \left[1 - \left(\frac{t_e}{t_0}\right)^{\frac{(1+3w)}{3+3w}}\right]$$

H/W Check:

In terms of H_0 & z:

$$\boxed{d_p(t_0) = \frac{c}{H_0}\,\frac{2}{1+3w}\left[1-(1+z)^{-(1+3w)/2}\right]}$$

~~prop.~~

Q What if $t_e = 0$? ← beginning of universe..

$$\Rightarrow d_p(t_0) = \int_0^{t_0} \frac{dt}{a(t)} \equiv d_{hor}(t_0)$$ ← Horizon distance.

↓

The "observable" universe.

$$\Rightarrow \boxed{d_{hor}(t_0) = t_0\,\frac{3(1+w)}{1+3w} = \frac{2}{H_0(1+3w)}}$$

H/W: Check d_{hor} for diff. values of w.

The ΛCDM model

Based on the available data:

→ Our universe has multiple components.

↓ Current!

$\Lambda \rightarrow$ dark energy.

CDM → Cold, Dark Matter.

→ In terms of density param. Ω_i:

the Friedmann eq:

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}$$

→ Composition:

$\Omega_\Lambda \longrightarrow \approx 69\%$

$\Omega_r \longrightarrow \sim 0.009\%$

$\Omega_m \longrightarrow \sim 4.8\%$

$\Omega_{dm} = 1 - \Omega_0 \Rightarrow \sim 26\%$