The Physics of Inflation

→ The Action:

$$S = \int d^4x \sqrt{-g}\,[R + \mathcal{L}_m]$$

$$\mathcal{L}_m = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - V(\phi)$$

→ Homogeneous & isotropic early universe,

$$\Rightarrow \quad \phi(x^\mu) \sim \phi(t)$$

$$\Rightarrow \quad \mathcal{L}_m \simeq \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

→ We call $\phi(x)$ the $\boxed{\text{inflaton}}$.
↓
drives inflation.

$$\therefore \quad \rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$$

$$P_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

→ Recall, during inflation $P = \omega\rho$, $\omega < -\frac{1}{3}$
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Inflaton ~ dark energy.

→ Inflation must continue for sufficiently long time to resolve Horizon, flatness & monopole problems.

⇒ Inflaton must vary sufficiently slowly.

i.e. $\dot{\phi}^2 << V(\phi)$

$$\Rightarrow \quad \rho_\varphi \approx P_\varphi \approx V(\varphi) \longleftrightarrow \text{Cosmological const.}$$

→ This slowly ~~a~~ changing mechanism of inflation is called slow-roll.

(Q) Under what conditions, the slow-rolling inflation is possible?

$$\dot{\rho}_\varphi + 3H(\rho_\varphi + P_\varphi) = 0$$

$$\Rightarrow \quad \dot{\varphi}\ddot{\varphi} + V'(\varphi)\dot{\varphi} + 3H\left(\tfrac{1}{2}\dot{\varphi}^2 + \cancel{V(\varphi)} + \tfrac{1}{2}\dot{\varphi}^2 - \cancel{V(\varphi)}\right) = 0$$

$$\Rightarrow \quad \cancel{\dot{\varphi}}\ddot{\varphi} + V'(\varphi)\cancel{\dot{\varphi}} + 3H\dot{\varphi}^{\cancel{2}} = 0$$

$$\Rightarrow \quad \boxed{\ddot{\varphi} + 3H\dot{\varphi} = -\frac{dV}{d\varphi}}$$

→ Jing friction force that approaches zero as $V(\varphi) \rightarrow$ minima.

→ For slow-rolling inflation:

$$\dot{\varphi} \ll V(\varphi) \Rightarrow \ddot{\varphi} << \dot{\varphi} \text{ or } \ddot{\varphi} \approx 0$$

$$\Rightarrow \quad 3H\dot{\varphi} = -\frac{dV}{d\varphi} \Rightarrow \boxed{\dot{\varphi} = -\frac{V'(\varphi)}{3H}}$$

Condition for slow-roll.

Let, $\dot{\varphi} = -\frac{1}{3H}\frac{dV}{d\varphi}$,

Then, $\dot{\varphi}^2 << V(\varphi)$

$$\Rightarrow \frac{1}{9H^2}\left(\frac{dV}{d\varphi}\right)^2 << V(\varphi)$$

$$\Rightarrow \boxed{V'(\varphi)^2 << 9H^2 V(\varphi)}$$ ← Slow-roll condition for inflaton potential

Q The Hubble parameter during inflation?
[Recall: $H(t) \simeq H_0$ ← const. for inflation].

$$H^2 = \frac{8\pi G}{3}\rho_\varphi$$

$$\Rightarrow \boxed{H_i \approx \sqrt{\frac{8\pi G V}{3}}}$$

→ Substituting in the potential condition:

$$V'^2 << \cancel{9}^{3} \cdot \frac{8\pi G}{\cancel{3}} V \cdot V$$

$$\Rightarrow \boxed{V'^2 << 24\pi G V^2}$$

$$\Rightarrow \boxed{\left(\frac{E_p}{V}\frac{dV}{d\varphi}\right)^2 << 1}$$; $\boxed{E_p \equiv \frac{1}{24\pi G}}$ ← Planck energy

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→ We have established <u>two</u> condition for slow-rolling inflation:

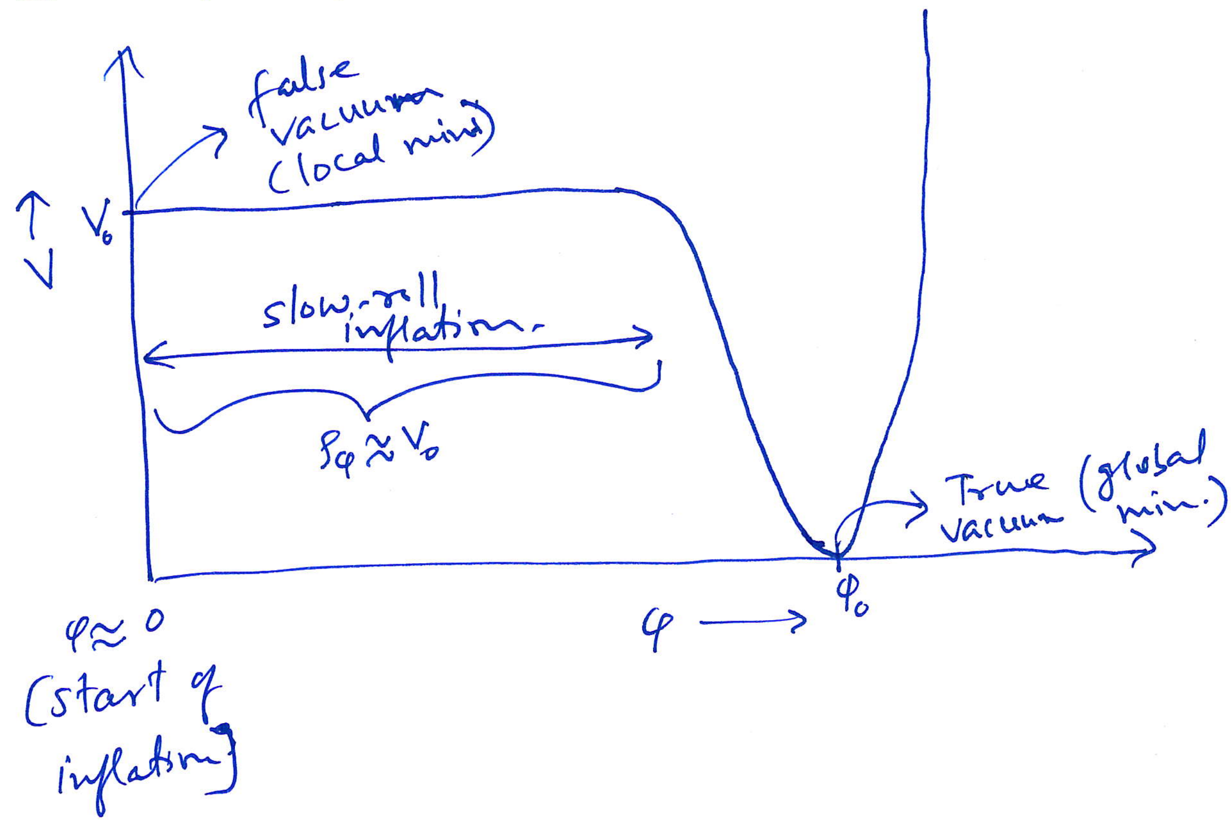
(1) Amplitude of potential should be sufficiently large to dominate the energy density of universe.

$$\rho_\varphi \approx V(\varphi)$$

(2) Gradient (slope) of inflaton pot. must be sufficiently shallow

$$\left(\epsilon_p \frac{V'}{V}\right)^2 << 1$$

<u>Example of $V(\varphi)$:</u>



→ In terms of the conformal time:

$$\eta = \int \frac{dt}{a(t)} = \int \frac{dt}{a_e e^{Ht}} = -\frac{1}{H}\frac{1}{a_e e^{Ht}}$$

$$\approx -\frac{1}{aH}$$

$$\downarrow$$

$$\int \frac{\frac{dt}{da}\cdot da}{a}$$

$$\downarrow$$

$$\int \frac{da}{a^2 H}$$

→ Eq. of inflaton field:

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0.$$

$$\therefore \quad \frac{d\varphi}{dt} = \frac{d\eta}{dt}\frac{d\varphi}{d\eta} = \frac{1}{a^2 H}\dot{a}\varphi' = \frac{1}{a}\frac{d\varphi}{d\eta} = \frac{1}{a}\varphi'$$

$$\frac{d^2\varphi}{dt^2} = \frac{d\eta}{dt}\frac{d}{d\eta}\left(\frac{d\varphi}{dt}\right) = \frac{1}{a}\frac{d}{d\eta}\left(\frac{1}{a}\varphi'\right)$$

$$= \frac{1}{a^2}\varphi''$$

$$\therefore \quad \frac{1}{a^2}\varphi'' + \frac{3H}{a}\varphi' + V' = 0$$

$$\Rightarrow \quad \boxed{\varphi'' + 3aH\varphi' + a^2 V' = 0}$$

→ Slow roll parameters:

– Quantify the "slow"-ness of inflaton field.

$$\epsilon \equiv \frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{H'}{aH^2} \rightsquigarrow \frac{dH}{d\eta}$$

⇒ during slow-roll, $\epsilon < 1$

→ Second slow-roll param.

$$\delta = \frac{1}{H}\frac{\ddot{\varphi}}{\dot{\varphi}} = -\frac{1}{aH\varphi'}\left[aH\varphi' - \varphi''\right]$$

$$= -\frac{1}{aH\varphi'}\left[3aH\varphi' + a^2 V_{,\varphi}\right]$$