Towards GTR

So far :

→ Due to equivalence principle, the geometry of space in presence of gravity cannot be Euclidean !! ← Non-flat !!

→ Regardless of gravity, the geometry of space and time is not Galilean / Euclidean due to the modified principle of relativity [!special relativity!]

→ Time is not universal

→ speed of light is universal.

$\Downarrow$

$\exists$ a new geometry of "spacetime" where time & space are treated on equal footing.

↓

Minkowski spacetime

→ Now, getting back to the Einstein's realization on equivalence principle :

Q How to describe this "non-flat"/curved geometry of spacetime under gravity that is also consistent with special relativity ?

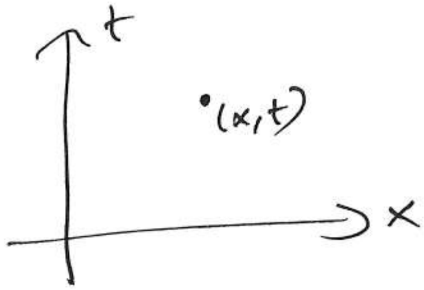
Ans: GTR ! BUT we first develop a formalism to consistently describe STR geometry.

Review of Tensors

~~The metric tensor:~~

The Einstein notations:

→ We recall that in Minkowski geometry time is also a coordinate label for an event.



→ There fore, an intuitive notation for an event:

$$u^\alpha = (t, x, y, z) \longrightarrow \text{similar to a 3-vector } u^i = (x, y, z) \equiv \vec{u}$$

→ Clearly, to find the norm, one has to compute $\sim \sum_\alpha u^\alpha u^\alpha$ (||er to $u^i u^i$ in ~~Galilean~~ Euclidean geom.)

→ Let's assume, that the repitition of indices implies a summation.

$$\Rightarrow \sum_\alpha u^\alpha u^\alpha \longrightarrow u^\alpha u^\alpha$$

→ An immediate observation:

How would one obtain an invariant distance (||er to 3D displacement: $u^i u^i$)?

Using the assignment for u^α above:

$$\cancel{u^\alpha u^\alpha} = u^\alpha \equiv \underbrace{t\hat{t} + x\hat{x} + y\hat{y} + z\hat{z}}_{\boxed{\text{our hypothesis}}}\ *$$

$$\Rightarrow \underbrace{u^\alpha u^\alpha = t^2 + x^2 + y^2 + z^2}_{\text{WRONG!!}},$$ assuming that $\hat{t}\cdot\hat{t} = 1 = \hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z}$

$\because$ We know, $-t^2 + x^2 + y^2 + z^2 = s^2$ in Minkowski geometry.

$$\Rightarrow s^2 = \cancel{[\eta] u^\alpha u^\alpha}\ [\eta][u][u] = -t^2 + x^2 + y^2 + z^2$$

$$\equiv \eta_{\alpha\beta} u^\alpha u^\beta$$

with $\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$ ← Metric

→ Definition of a tensor: ⊛ Skipping technicalities for a later course! (not within current scope)

A tensor of type $\binom{0}{N}$ is a function of N vectors into the real numbers, which is linear in each of its N arguments.

eg.: $g(\vec{A}, \vec{B}) \equiv \vec{A}\cdot\vec{B}$ ← $\binom{0}{2}$ tensor.

$f(t, x, y, z)$ ← $\binom{0}{0}$ tensor.

* For some unit vectors $\vec{e}_{\alpha,\beta}$:

$$g(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta \equiv \eta_{\alpha\beta}$$ ← $\boxed{\text{Metric tensor}}$

→ |||ly a tensor

$$\underbrace{p_\alpha}_{\text{1-form}} \equiv \tilde{p}(\vec{e}_\alpha) \leftarrow \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{tensor}}$$

$\boxed{\text{1-form}}$

where, $\vec{e}_\alpha$ is a unit vector.

* Note that the position of index is as subscript for 1-form.

$$\Rightarrow \begin{pmatrix} 0 \\ N \end{pmatrix} \text{ tensor : } \quad \underbrace{p_{\alpha\beta\ldots\gamma} \equiv \tilde{p}(\vec{e}_\alpha, \vec{e}_\beta, \ldots \vec{e}_\gamma)}_{\boxed{\text{N-form}}}$$

Q Based on this, what can you say about the decomposition a rank-2 antisymmetric field $B_{\mu\nu}$ into scalar/vector/tensor components?

→ Now, for a vector $\vec{A} \equiv A^\alpha \vec{e}_\alpha$

$$\Rightarrow \tilde{p}(\vec{A}) = \tilde{p}(A^\alpha \vec{e}_\alpha) = A^\alpha \tilde{p}(\vec{e}_\alpha)$$

$$\Rightarrow \boxed{\tilde{p}(\vec{A}) = A^\alpha p_\alpha} \leftarrow \text{1-form for a gen. vector } \vec{A}.$$

⇒ A 1-form is different from a 4-vector!

Q Relation between a 1-form (A_α?) and corresponding vector A^α?

→ Since the metric is a mapping given by:

$$g(\vec{A}, \vec{B}) \equiv \vec{A} \cdot \vec{B}$$

~~For a 1-form:~~

$$\cancel{\tilde{p}(\vec{A})}$$

Suppose $g(\vec{V}, \vec{A}) = \vec{V} \cdot \vec{A}$

Let's define a 1-form $\tilde{V}$:

$$\boxed{\tilde{V}(\vec{A}) \equiv \vec{V} \cdot \vec{A}}$$

($\because$ $\vec{V}$ is permanent but $\vec{A}$ is the argument
⇓
satisfies the $\binom{0}{1}$ def.)

⇒ The components of such a 1-form:

$$V_\alpha \equiv \tilde{V}(\vec{e}_\alpha) = \vec{V} \cdot \vec{e}_\alpha = \vec{e}_\alpha \cdot \vec{V}$$

$$= \vec{e}_\alpha \cdot V^\beta \vec{e}_\beta$$

$$\Rightarrow \quad V_\alpha = (\vec{e}_\alpha \cdot \vec{e}_\beta) V^\beta$$

$$\Rightarrow \quad \boxed{V_\alpha \equiv \eta_{\alpha\beta} V^\beta}$$ ← Finally looks familiar!!

In physics texts :

V_α → covariant 4-vector ← "1-form"

V^α → contravariant 4-vector.

* **Note**: Important to distinguish 1-form and vector due to the difference in gradient components or just components.

For example:

Consider a vector $V^\alpha : (a, b, c, d)$

the difference b/w 1-form & vec.

~~$\Rightarrow V_\beta = (-a, b, c, d$~~

$$\Rightarrow V_\beta = \eta_{\beta\alpha} V^\alpha = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$= \boxed{(-a, b, c, d)}$$

Ily,

$$\left.\begin{aligned} \tilde{d}\phi &\rightarrow \left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \ldots\right) \\ \tilde{\partial}\phi &\rightarrow \left(-\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \ldots\right) \end{aligned}\right\} \text{how?}$$

→ To understand the above, we will now review the tensor calculus.

* If an inverse of the metric exists, i.e.:

$$(\eta_{\alpha\beta})^{-1} = \eta^{\alpha\beta} \leftarrow \text{Non-vanishing determinant.}$$

then, $\boxed{A^\alpha = \eta^{\alpha\beta} A_\beta}$

→ Magnitude of a vector & 1-form:

$$\cancel{\tilde{p}^2 = \eta_{\alpha\beta} p^\alpha p^\beta = \vec{p}^2}$$

$$\cancel{\tilde{p}^2 = \eta_{\alpha\beta}}$$

$$\vec{p}^2 = \eta_{\alpha\beta} p^\alpha p^\beta$$

$$= \eta_{\alpha\beta} \left(\eta^{\alpha\mu} p_\mu\right) \left(\eta^{\beta\nu} p_\nu\right)$$

$$= \eta^{\alpha\mu} p_\mu \left(\eta_{\alpha\beta} \eta^{\beta\nu}\right) p_\nu$$

Inverse property.

$$= \eta^{\alpha\mu} p_\mu \delta_\alpha{}^\nu p_\nu$$

$$\Rightarrow \boxed{\vec{p}^2 = \eta^{\alpha\mu} p_\alpha p_\mu = \tilde{p}^2}$$

For general vec. : $p_\mu q^\mu$ → scalar product.

→ Raising & lowering indices:

Easy to infer that going from co- to contra- is through the use of metric.

$$p_\mu = \eta_{\mu\nu} p^\nu \quad \& \quad p^\mu = \eta^{\mu\nu} p_\nu$$

Q For the flat Minkowski metric $\eta_{\alpha\beta}$, what is $\eta^\alpha{}_\beta$?

Differentiation & Calculus of tensors.
(without the baggage of mathematics!)

$$\frac{\partial p^{\nu}}{\partial x^{\alpha}} \equiv p^{\nu}{}_{,\alpha}$$

$$\frac{\partial p_{\nu}}{\partial x^{\alpha}} \equiv p_{\nu,\alpha} = \frac{\partial}{\partial x^{\alpha}}\left(\eta_{\nu\mu} p^{\mu}\right) = \eta_{\nu\mu} \frac{\partial p^{\mu}}{\partial x^{\alpha}}$$

iff $\eta_{\mu\nu} \rightarrow$ Minkowski

In general curved manifold:

$$\frac{\partial p_{\nu}}{\partial x^{\alpha}} = \frac{\partial}{\partial x^{\alpha}}\left(g_{\nu\mu} p^{\mu}\right) = g_{\mu\nu,\alpha} p^{\mu} + g_{\mu\nu} p^{\mu}{}_{,\alpha}$$

Now, Recall – Back to Physics:

We have figured out the mathematical framework to describe Minkowski spacetime of STR, using tensors (3+1D) & Minkowski metric.

→ Now, we must introduce curvature in STR that is consistent with equivalence principle.

H/W [Read Chap 5.1, Schutz]

→ Local Lorentz frame and equivalence principle.