# (Not so) Brief Review of GTR

In STR: $\cancel{ds^2}\ S^2 = \underbrace{\eta_{\mu\nu}}\, x^\mu x^\nu$

$$\eta_{\mu\nu} \xrightarrow{\text{what if}} g_{\mu\nu}(x)$$

→ The invariant length element:

$$ds^2 = \underbrace{g_{\mu\nu}(x)}\, dx^\mu dx^\nu$$

↓ describes a spacetime geometry different from Minkowski.

→ A general curved manifold is called "Reimannian".

→ Equivalence principle: Locally, the manifold is Minkowski so that "local Lorentz transformation" ~~is~~ connects the coordinate frames.

Cf. Prof: p149 - Schwartz -

$$g_{\mu\nu}(x)\, dx^\mu dx^\nu = g_{\alpha\beta}(x')\, dx'^\alpha dx'^\beta$$

$$= g_{\alpha\beta}(x') \frac{\partial x'^\alpha}{\partial x^\rho} dx^\rho \frac{\partial x'^\beta}{\partial x^\sigma} dx^\sigma$$

$$= \left( g_{\alpha\beta}(x') \frac{\partial x'^\alpha}{\partial x^\rho} \frac{\partial x'^\beta}{\partial x^\sigma} \right) dx^\rho dx^\sigma$$

$$\Rightarrow \boxed{g_{\mu\nu}(x) = g_{\alpha\beta}(x') \frac{\partial x'^\alpha}{\partial x^\rho} \frac{\partial x'^\beta}{\partial x^\sigma}} \longleftarrow \text{generalizing } \Lambda^\alpha{}_\rho \,!!$$

For consistency in notations, let's denote

$$\frac{\partial x'^{\alpha'}}{\partial x^{\beta}} \equiv \Lambda^{\alpha'}{}_{\beta}$$

→ It is easy to show that

$$\cancel{\partial x^{\beta}}\ \frac{\partial x^{\alpha}}{\partial x'^{\beta'}} \equiv \boxed{\Lambda^{\alpha}{}_{\beta'} = \left(\Lambda^{\alpha'}{}_{\beta}\right)^{-1}}\,,$$ H/W

~~H/W~~ * The space (Manifold) in which this general metric $g_{\mu\nu}(x)$ is defined is called a "Riemannian Manifold".

H/W Show that for a symmetric matrix A with components $\in \mathbb{R}$, $\exists$ matrix H s.t. $H^T A H$ is diagonal with eigenvalues of A.

→ Using the above theorem:

For a $g_{\alpha'\beta'}(x)$, $\exists\ \Lambda^{\alpha'}{}_{\beta}$ s.t.

$$\Lambda^{\alpha'}{}_{\beta}\, g_{\alpha'\beta'}\, \Lambda^{\beta'}{}_{\alpha} = \eta_{\beta\alpha} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

→ <u>Physical implication:</u>

The above result implies, that at every "local" point in spacetime, $x^\mu$, with a global metric $g_{\mu\nu}(x)$ it is possible to define (transform) a Minkowski metric valid in the neighbourhood of $x^\mu$.

⇒ "The local flatness"

→ The corresponding ~~of~~ local frame is called the "<u>local Lorentz frame</u>".

→ <u>Locally flat coordinate system:</u>

[skipping mathematical details outside the scope of this course; Read <u>Schutz Sec. 6.2, p149</u>]

→ Given any point P on the Riemannian manifold, a coordinate system $x^\alpha$ can be found whose <u>origin is at P</u> and:

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} + O(x^\mu x_\mu)$$

Or

$$g_{\alpha\beta}(P) = 0 \quad ; \quad \frac{\partial}{\partial x^\gamma} g_{\alpha\beta}(P) = 0 \quad \forall \alpha, \beta, \gamma$$

But, $\frac{\partial^2}{\partial x^\gamma \partial x^\mu} g_{\alpha\beta}(P) \neq 0$ in general.

↓

~~"Weak~~ "Condition for local flatness"

→ ## The principles of General Relativity:

(I) Spacetime, which is a set consisting of all events, is a four-dimensional manifold whose geometry is described by a metric.

(II) The metric is measurable by rods and clocks. The distance along a rod between two nearby points is $\boxed{|d\vec{x}\cdot d\vec{x}|^{1/2}}$ and the time measured by a clock that experiences two events closely separated in time is $\boxed{|-d\vec{x}\cdot d\vec{x}|^{1/2}}$

→ timelike ($ds^2<0$)

spacelike ($ds^2>0$)

→ There do not exist (global) coordinates in which $d\vec{x}\cdot d\vec{x} = ds^2 = -dt^2 + dx^{i\,2}$ everywhere.

(III) The metric of spacetime can be put in the Lorentz form $\eta_{\alpha\beta}$ at any particular event by an appropriate choice of coordinates.

→ The above three statements/principles specify the geometry.

[Q] How do particles/fields/fluids behave in this "curved" geometry.

## IV The Weak equivalence principle:

"Freely falling particles move on timelike geodesics of the spacetime"

No other interactions except gravity.

Q How about fluids?

## V Einstein equivalence principle — a generalization of weak equivalence principle.

"Any local physical experiment not involving gravity will have the same result if performed in a freely falling inertial frame as if it were performed in the flat spacetime of STR."

Does not involve fields.

The statement of the local Lorentz frame.

→ Next, we look at implication of GR.

→ The covariant differentiation.

→ In a general Riemannian manifold, the concepts of derivatives, translation, etc. get difficult to describe.

→ Fortunately we have a connection to the Minkowski spacetime (at least locally) to help our understanding.

→ Consider, a local Lorentz frame at point $P$ & a vector $V^\alpha$ :

$$\frac{\partial V^\alpha}{\partial x^\beta} \equiv V^\alpha_{,\beta} \leftarrow \text{Our notation for derivatives.}$$

Illy, consider the derivative of metric in this frame:

$$g_{\mu\nu,\alpha}(P) = \boxed{\eta_{\mu\nu,\alpha} = 0} \leftarrow \text{identically zero!!}$$

⇒ Because this a tensor equation, and since at each point $P$ in the global manifold with metric $g_{\alpha\beta}$ ∃ a local frame with metric $\eta_{\alpha\beta}$, it must be possible to generalize this "locally valid" equation to be "globally true" or "covariant" !!

→ However, in general

$$\frac{\partial}{\partial x^{\beta}} \frac{\partial}{\partial x^{\mu}} g_{\alpha\beta}(x) \neq 0 \ !!$$

⇒ We define a so-called "covariant derivative":
such that,

$$\nabla_{\mu} g_{\alpha\beta} \equiv \boxed{g_{\alpha\beta;\mu} = 0} \text{ for all basis.}$$

→ With appropriate tensor calculus, (which we skip due to shortage of time in this course), [Read Chap. 5 from Schutz]

it can be shown that:

$$\boxed{g_{\alpha\beta;\mu} \equiv g_{\alpha\beta,\mu} - \Gamma^{\nu}{}_{\alpha\mu} g_{\nu\beta} - \Gamma^{\nu}{}_{\beta\mu} g_{\alpha\nu}}$$

where, new quantities "$\Gamma$" called the "Christoffel connections" are introduced.

→ The Christoffel connections:

$$\boxed{\Gamma^{\alpha}{}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta} \left( g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta} \right)}$$

$$\rightarrow \boxed{\Gamma^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\nu\mu}}$$

H/W How do $\Gamma^{\alpha}{}_{\mu\nu}$ transform under GCT? ~~Is it a~~ Does it transform like a tensor?