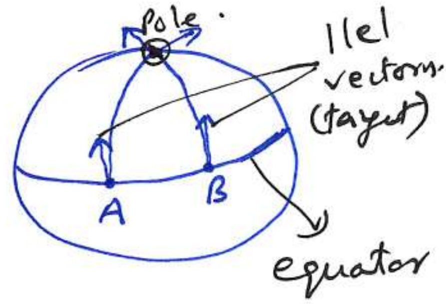
Q Why do we need to change the notion of a derivative in a curved spacetime?

→ Parallel transport — A demonstration of the inadequacy of std. derivative!

$\boxed{\text{Case - I:}}$ Two parallel lines never intersect in flat space.

→ But on a sphere, two parallel vectors along the equator will intersect at the pole!!

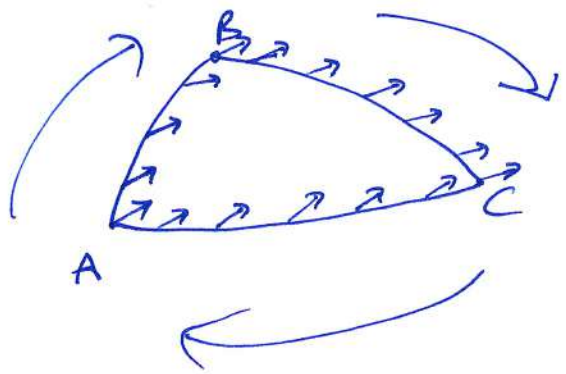


$\Downarrow$

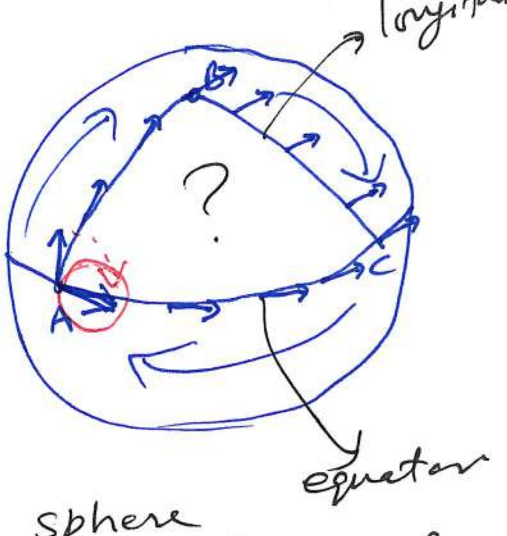
$\boxed{\text{Not flat}}$

→ Called the "intrinsic curvature" of a (n-1)D space enclosed in n-D manifold.

$\boxed{\text{Case - II:}}$ Parallel transport of a vector along a closed path.



Flat space
(Intrinsically flat)

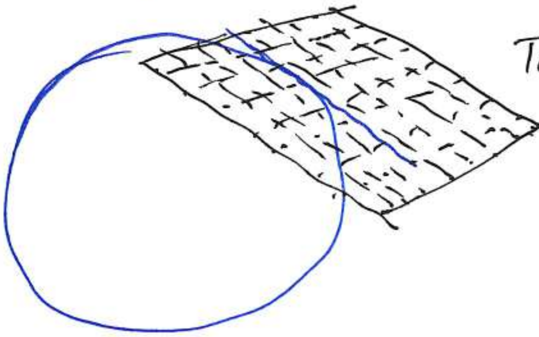


sphere
(intrinsically curved 2D surface.)
embedded in 3D space.

→ We note that :

* In curved space, the result of parallel transporting a vector from one point to another depends on the path taken.

* Two vectors can only be compared in a natural way if they are elements of the same tangent space.



Tangent surface to a pt. on a curved sphere.

* So, two particles ~~cros~~ passing by ~~each~~ close to each other can have well defined rel. vel. but eg. two galaxies moving wrt. eachother can't have a ~~so~~ well defined rel. velocity

→ ∴ The perceived redshift of galaxies is not because they are $\boxed{\text{receding}}$, but because the universe is expanding!

In fact we can't comment on it since rel. vel. can't be defined.

Parallel transport in a curved spacetime.

→ Consider an arbitrary tensor $T^{\mu_1\mu_2\dots\mu_k}{}_{\nu_1\dots\nu_l}$ to be transported along a curve $x^\mu(\lambda)$.

→ The requirement for ∥el transport, in flat space:

$$\boxed{\frac{d}{d\lambda}T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l} = 0}$$

→ ~~To g~~ Note: The above tensor eq. is a diff. eq. in flat spacetime, and we need to generalize this eq. (covariantly) to curved spacetime!!

⇒ Using the covariant derivative, one can generaline parallel transport eq. to curved manifold.

[Recall: $\cancel{g_{\mu\nu,\alpha}}$ $\eta_{\mu\nu,\alpha} = 0 \longrightarrow g_{\mu\nu;\alpha} = 0$]

$$\Rightarrow \frac{d}{d\lambda}T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l} \longrightarrow \frac{D}{d\lambda}T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l} = \frac{dx^\mu}{d\lambda}\nabla_\mu T^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l}$$

where, we have defined a directional covariant derivative:

$$\boxed{\frac{D}{d\lambda} \equiv \frac{dx^\mu}{d\lambda}\nabla_\mu}$$

$$\Rightarrow \boxed{\frac{dx^\sigma}{d\lambda} \nabla_\sigma T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = 0} \leftarrow$$ Equation of parallel transport.

Case: A vector V^μ:

Then, $$\nabla_\sigma V^\mu = \frac{\partial}{\partial x^\sigma} V^\mu + \Gamma^\mu_{\sigma\rho} V^\rho$$

$$\Rightarrow \frac{dx^\sigma}{d\lambda} \nabla_\sigma V^\mu = \boxed{\frac{d}{d\lambda} V^\mu + \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} V^\rho = 0}$$

Eq. of $||$el transport for a vector!!

Case: The metric $g_{\mu\nu}$:

[Q] What is the $||$el transport of metric?

$$\frac{D}{d\lambda} g_{\mu\nu} = \boxed{\frac{dx^\sigma}{d\lambda} \nabla_\sigma g_{\mu\nu} = 0 \quad \forall\, g_{\mu\nu}}$$

Identically true!!

[H/W] Show that if two vectors V^μ & W^ν are $||$el transported, then their inner product (scalar product / contraction) is preserved.

i.e $\frac{d}{d\lambda}(V_\mu W^\mu) = 0$

Next step: Generalizing the straight line to curved manifold!

$\downarrow$

The path of shortest distance between two points.

$\updownarrow$

$\boxed{\text{A path that parallel-transports its own tangent vector.}}$

$\rightarrow$ The tangent vector to a path $x^\mu(\lambda)$:

$$\boxed{v^\mu \equiv \frac{dx^\mu}{d\lambda}}$$

$\Rightarrow$ Parallel transport eq. of tangent vector:

$$\frac{D}{d\lambda} v^\mu = \boxed{\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0}$$

$$\Rightarrow \frac{dx^\rho}{d\lambda} \nabla_\rho \left(\frac{dx^\mu}{d\lambda}\right) = 0$$

$$\Rightarrow \frac{\cancel{dx^\rho}}{d\lambda} \frac{\partial}{\cancel{\partial x^\rho}} \left(\frac{dx^\mu}{d\lambda}\right) + \Gamma^\mu_{\rho\sigma} \left(\frac{dx^\sigma}{d\lambda}\right)\left(\frac{dx^\rho}{d\lambda}\right) = 0$$

$$\Rightarrow \boxed{\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\sigma}{d\lambda}\frac{dx^\rho}{d\lambda} = 0}$$

$\uparrow$

$\boxed{\text{Geodesic equation.}}$

(*) The curve $x^\mu(\lambda)$ is called the "geodesic".

$\boxed{\text{H/W}}$ Solve the geodesic equation for Minkowski spacetime.

→ An alternative derivation (understanding) of the geodesic equation:

How do we define distance in Riemannian spacetime?

→ The invariant length element!

$$d\tau^2 = g_{\mu\nu}\, dx^\mu dx^\nu = dx_\mu dx^\mu$$

$$\Rightarrow \quad d\tau = \sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}$$

$$\Rightarrow \quad \int d\tau = \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

$$\Rightarrow \quad \tau = \int d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

~~Fo~~ Consider a time-like trajectory.

$\Rightarrow \tau \rightarrow$ proper time.

& $\tau^2 < 0$

$$\Rightarrow \quad \tau = \int d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

Q What is the extremum path (trajectory) of a "test" particle represented by the dynamical variable $x^\mu(\tau)$?

Sound familiar !?
[Least Action principle!!]

$\therefore$ ~~The~~ Considering this "action" of a free particle

$$\tau = \int d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

If, we replace $\lambda \longrightarrow \tau$ (proper time)

$$\Rightarrow \quad \tau = \int d\tau \sqrt{\underbrace{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}}$$

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = g_{\mu\nu} u^\mu u^\nu$$

$$= \underbrace{u_\mu u^\mu}$$

$\boxed{\text{4-velocity}}$

Now,

$$\because \; dx_\mu dx^\mu = d\tau^2 \quad \sigma < 0$$

$$\Rightarrow \frac{dx_\mu}{d\tau} \frac{dx^\mu}{d\tau} = -1 \Longleftrightarrow \boxed{u_\mu u^\mu = -1}$$

→ Coming back to the integral,
Using the variational principle :

$$\delta \tau = \int d\tau \, \delta \sqrt{\underbrace{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}_{u_\mu u^\mu \, \equiv \, f}}$$

$u_\mu u^\mu$ ← Scalar !!

$$\therefore \quad \delta \sqrt{-f} = \frac{1}{2} (-f)^{-1/2} \, \delta f$$

$$\Rightarrow \delta \sqrt{-g_{\mu\nu} u^\mu u^\nu} = \frac{1}{2} \cancelto{1}{\frac{1}{\sqrt{-u_\mu u^\mu}}} \, \delta \left(g_{\mu\nu} u^\mu u^\nu \right)$$

Now,

$$\delta \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) = \left(\delta g_{\mu\nu}(x) \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{d}{d\tau} \delta x^\mu \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau} (\delta x^\nu)$$

Here,

$$\delta g_{\mu\nu}(x) = g_{\mu\nu,\rho} \, \delta x^\rho$$

~~→ Re-writing τ as the Action of free particle :~~

$$\cancel{S = \frac{1}{2} \int d\tau}$$

→ Therefore, substituting in $\delta\tau$:

$$\delta\tau = \frac{1}{2}\int d\tau\, \delta\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) = 0 \quad \text{[variational principle]}$$

$$\Rightarrow \frac{1}{2}\int d\tau \Big[\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\rho + \underbrace{g_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{d}{d\tau}\delta x^\mu}$$
$$+ \underbrace{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d}{d\tau}\delta x^\nu}\Big] = 0$$

$$\cancel{\Rightarrow \frac{1}{2}\int d\tau \Big[}$$

→ Dealing with $\frac{d}{d\tau}(\delta x^\mu)$:

$$\int d\tau\, g_{\mu\nu}\frac{dx^\nu}{d\tau}\frac{d}{d\tau}(\delta x^\mu) = -\int d\tau \frac{d}{d\tau}\left[g_{\mu\nu}\frac{dx^\nu}{d\tau}\right]\delta x^\mu$$
$$= -\int d\tau \left[\frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\right]\delta x^\mu$$

$$\Rightarrow \delta\tau = \frac{1}{2}\int d\tau \Big[g_{\mu\nu,\rho}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\rho - \left(g_{\mu\nu,\rho}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\right)\delta x^\mu$$
$$- \left(g_{\mu\nu,\rho}\frac{dx^\rho}{d\tau}\frac{dx^\mu}{d\tau} + g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\right)\delta x^\nu\Big] = 0$$

$$\Rightarrow \frac{1}{2}\int d\tau \Big[2g_{\mu\rho}\frac{d^2x^\mu}{d\tau^2} + (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Big]\delta x^\rho = 0$$

$\cancel{\int d\tau}$ [Q] Where did the boundary terms go?
$\rightarrow$ "Cauchy boundary conditions".

∴ From the least action principle:

$$\Rightarrow g_{\mu\rho}\frac{d^2x^\mu}{d\tau^2}+\underbrace{\frac{1}{2}\left(\partial_\mu g_{\nu\rho}+\partial_\nu g_{\rho\mu}-\partial_\rho g_{\mu\nu}\right)}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0$$

$$\Downarrow\ ?$$

$$\Rightarrow g^{\rho\sigma}g_{\mu\rho}\frac{d^2x^\mu}{d\tau^2}+\frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\rho}+\partial_\nu g_{\rho\mu}-\partial_\rho g_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0$$

$$\Rightarrow \delta_\mu^\sigma\frac{d^2x^\mu}{d\tau^2}+\underbrace{\frac{1}{2}g^{\rho\sigma}\left(g_{\nu\rho,\mu}+g_{\rho\mu,\nu}-g_{\mu\nu,\rho}\right)}_{\Downarrow}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0$$

$$\Gamma^\sigma_{\nu\mu}=\Gamma^\sigma_{\mu\nu}$$

[Note: We have <u>derived</u> $\Gamma^\sigma_{\mu\nu}$ from first principles!!]

∴

$$\Rightarrow \boxed{\frac{d^2x^\sigma}{d\tau^2}+\Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0}$$

$\rightarrow$ This is why the action principle is so fundamental (and powerful!) — We derived an essential feature of curved spacetime geometry without knowing diff. geom.!!

ONE MORE THING!!

Q How does the geodesic equation ~~evolve~~ change under the linear transformation:

$$\tau \longrightarrow \lambda \equiv a\tau + b, \quad a, b \text{ constant?}$$

$$\frac{d^2x^\mu}{d\tau^2} = \frac{d\lambda}{d\tau}\left(\frac{d}{d\lambda}\left(\frac{d\lambda}{d\tau}\frac{dx^\mu}{d\lambda}\right)\right) = a^2\frac{d^2x^\mu}{d\lambda^2}$$

$$\frac{dx^\mu}{d\tau} = \frac{d\lambda}{d\tau}\frac{dx^\mu}{d\lambda} = a\frac{dx^\mu}{d\tau}$$

$$\Rightarrow \boxed{a^2\left[\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}\right] = 0} \longleftarrow \boxed{\text{Invariant!!}}$$

→ Such parameters λ s.t. $\boxed{\lambda = a\tau + b}$ for which the geodesic eq. is invariant is called

affine parameter!!