The Curvature Tensors

→ Without going into the diff. geom. machinery to define curvature tensors, we introduce the Riemann curvature tensor, defined in terms of the Christoffel connections:

$$\boxed{R^{\rho}{}_{\sigma\mu\nu} \equiv \Gamma^{\rho}_{\nu\sigma,\mu} - \Gamma^{\rho}_{\mu\sigma,\nu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}}$$

NOTE:

→ $\boxed{R^{\rho}{}_{\sigma\mu\nu} = - R^{\rho}{}_{\sigma\nu\mu}}$

H/W: Show that $[\nabla_\mu, \nabla_\nu] V^\rho = R^{\rho}{}_{\sigma\mu\nu} V^\sigma - 2\Gamma^{\lambda}_{[\mu\nu]} \nabla_\lambda V^\rho$

NOTE:

→ Since, $R^{\rho}{}_{\sigma\mu\nu}$ is solely defined in terms of connections, it quantifies the curvature of a spacetime.

→ Flat spacetime (eg. Minkowski) $\Rightarrow \Gamma^{\lambda}_{\mu\nu} = 0$

$\Rightarrow R^{\rho}{}_{\sigma\mu\nu} = 0$

describes (measures) curvature ∵ there's $\partial_\mu \partial_\nu g_{\sigma\rho}$ terms!!

OR

If $R^{\rho}{}_{\sigma\mu\nu} = 0 \Rightarrow$ Flat spacetime.

Helps to check curvature of non-trivial spacetime.

H/w: Consider spherical polar coordinate system where $g_{\mu\nu} \neq \eta_{\mu\nu}$.

(1) Calculate Γ.

(2) Calculate $R^{\rho}{}_{\sigma\mu\nu}$.

Properties of $R^{\rho}{}_{\sigma\mu\nu}$:

(1) $R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}{}_{\sigma\mu\nu}$

~~(2) $R_{\rho\sigma\mu\nu} = =$~~

=

H/w In a local Lorentz frame at pt. p where $\Gamma^{\rho}_{\mu\nu}(p) = 0$ & $\partial_\mu g^{\lambda\tau} = 0$, but $\partial_\mu \partial_\nu g^{\lambda\tau} \neq 0$.

$$\therefore R_{\hat\rho\hat\sigma\hat\mu\hat\nu}(p) = g_{\hat\rho\hat\lambda}\left(\partial_{\hat\mu}\Gamma^{\hat\lambda}_{\hat\nu\hat\sigma} - \partial_{\hat\nu}\Gamma^{\hat\lambda}_{\hat\mu\hat\sigma}\right)$$

$$= \frac{1}{2}\left(\partial_{\hat\mu}\partial_{\hat\sigma} g_{\hat\rho\hat\nu} - \partial_{\hat\mu}\partial_{\hat\rho} g_{\hat\nu\hat\sigma} - \partial_{\hat\nu}\partial_{\hat\sigma} g_{\hat\rho\hat\mu} + \partial_{\hat\nu}\partial_{\hat\rho} g_{\hat\mu\hat\sigma}\right)$$

$$\Rightarrow \boxed{R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}} \quad \& \quad \boxed{R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}}$$

$$\Rightarrow \boxed{R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}}$$

$$\& \quad \boxed{R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0}$$

$\text{IIly}\quad \nabla_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} + \nabla_{\hat{\rho}} R_{\hat{\sigma}\hat{\lambda}\hat{\mu}\hat{\nu}} + \nabla_{\hat{\sigma}} R_{\hat{\lambda}\hat{\rho}\hat{\mu}\hat{\nu}}$

$$\equiv \boxed{\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0} \longleftarrow \text{Bianchi identity}$$

Ricci tensor: Rank-2 tensor constructed from the Riemann tensor:

$$\left.\begin{aligned} &\boxed{R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}} \\ \Rightarrow\quad &R_{\mu\nu} = R_{\nu\mu} \end{aligned}\right\} \boxed{\text{H/W: Compute.}}$$

Ricci scalar: Further contraction of indices to define a scalar out of curvature tensor:

$$\boxed{R \equiv g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu}}$$

→ ENOUGH GEOMETRY!!

Where's the physics?

What's the action of GR?

Action for $g_{\mu\nu}(x)$?

→ As we have seen with geodesics, the action is a powerful construct to describe physics.

$\boxed{Q}$ How to construct ~~an~~ the action for $g_{\mu\nu}(x)$ as a dynamical variable?

→ To construct the action : $S = \int d^4x \, \mathcal{L}$

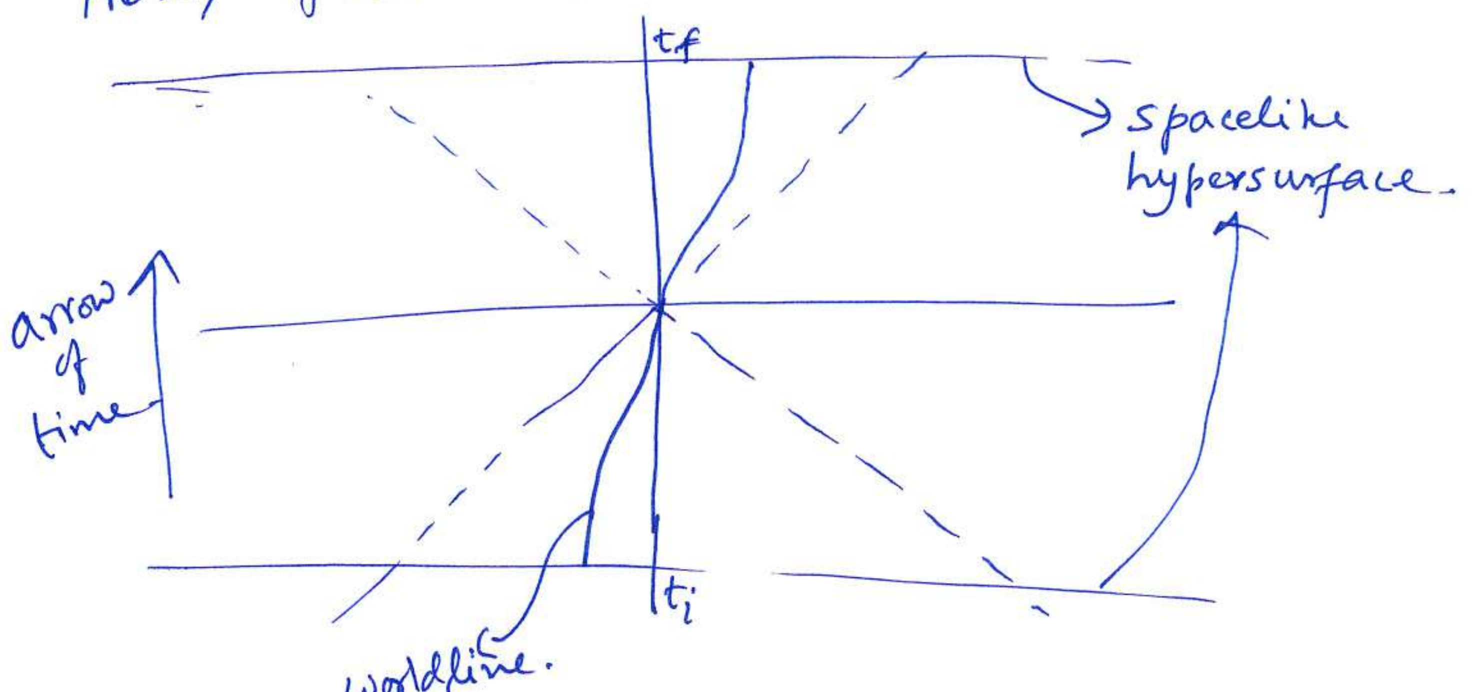
(1) Must be invariant. ← scalar, scalar products!

(2) Must be unit-less. ← $[\mathcal{L}] \sim [L]^{-4}$!

(3) Must be atleast quadratic order in the dynamical variable (metric) for nontrivial dynamics. ← $\sim (\partial_\mu g_{\rho\sigma})^2$!

→ $g_{\mu\nu}(x)$ is a field. If we are to construct an action
$$S = \int_{t_i}^{t_f} dt \, L$$

Here, first $t_i \to -\infty$, $t_f \to \infty$.



→ Since, fields span the entire space time and are in-principle unbounded,

$$L = \int d^3\vec{x}\, \underbrace{\mathcal{L}}_{\text{Lagrangian density}}$$

$$\Rightarrow \quad \boxed{S = \underbrace{\boxed{\int d^4x}}_{\text{Invariant?}} \underbrace{\boxed{\mathcal{L}}}_{\text{Invariant?}}}$$

Now,

(*) An ~~invariant~~ scalar S can be constructed if we can write an <u>invariant volume element</u>.

& an invariant $\mathcal{L}[g_{\mu\nu}]$.

$\Updownarrow$

$\boxed{\text{Lorentz scalar}}$

→ Consider : $d^4x = dx^0 dx^1 dx^2 dx^3$

Under GCT :

$$d^4x' = dx'^0\, dx'^1\, dx'^2\, dx'^3 = \boxed{\left(\frac{\partial x'^0}{\partial x^0}\frac{\partial x'^1}{\partial x^1}\frac{\partial x'^2}{\partial x^2}\frac{\partial x'^3}{\partial x^3}\right)} \times dx^0 dx^1 dx^2 dx^3$$

$$= \underbrace{\det\left(\frac{\partial x'^{\mu'}}{\partial x^{\mu}}\right)}_{\text{Jacobian.}} \equiv \det(\Lambda^{\mu'}{}_{\mu})$$

Now, we know:

$$g' \equiv \det(g'_{\mu'\nu'}) = \det\left(\frac{\partial x^{\mu}}{\partial x'^{\mu'}} g_{\mu\nu} \boxed{\frac{\partial x^{\nu}}{\partial x'^{\nu'}}} \right) \qquad J^{-1} \sim \Lambda^{-1}$$

$$= \det(g_{\mu\nu}) \cdot \det(\Lambda^{T})^{-1} \det(\Lambda)^{-1}$$

$$= g \cdot \left[\det(\Lambda)^{-1}\right]^{2}$$

∴ By defining the volume element as:

$$\sqrt{-g}\, d^4x \longrightarrow \sqrt{-g'}\, d^4x' = \sqrt{-g}\, \cancel{|\det(\Lambda)^{-1}|} \cdot \cancel{\det(\Lambda)}\, d^4x$$

$$\boxed{= \sqrt{-g}\, d^4x} \checkmark$$

Hence,

$$\boxed{S = \int d^4x \sqrt{-g}\, \mathcal{L}}$$

where $\mathcal{L} = \mathcal{L}[g_{\mu\nu}(x)]$ ← scalar constructed out of tensors involving quad. derivatives of $g_{\mu\nu}$.

Q How to construct a scalar that is:

(1) a function of $g_{\mu\nu}$.

(2) quadratic order of derivatives of $g_{\mu\nu}$?

Ans: We already know!! — The Ricci scalar

→ $\boxed{\mathcal{L} = R}$ ← The simplest action of gravity.

→ The Einstein Action.

$$\boxed{S = \int d^4x \sqrt{-g}\, R}$$