The Einstein Equation

[Derived from the Einstein-Hilbert action]

$$S_g \equiv S_{Grav} = \int \underbrace{d^4x\sqrt{-g}}_{\text{invariant vol. element}}\ \overbrace{R}^{\text{Ricci scalar}}$$

→ If there is matter (duh!),

$$S_m = \int d^4x\sqrt{-g}\,\mathcal{L}_m$$

$$\therefore \quad S_{tot} \equiv S = S_g + S_m = \int d^4x\sqrt{-g}\left[R + \mathcal{L}_m\right]$$

Warm up: Variating S_m:

Let, $\mathcal{L}_m = \mathcal{L}_m(\varphi(x), \partial_\mu\varphi(x))$, for some $\varphi(x)$.

Then, according to the least action principle,

$$\delta S_m = \int d^4x\,\delta\left(\sqrt{-g}\,\mathcal{L}_m(\varphi, \partial_\mu\varphi)\right) = 0$$

$$\cancel{\Rightarrow \int d^4x\left[\delta\sqrt{-g}\,\mathcal{L}_m + \sqrt{-g}\,\delta\mathcal{L}_m\right] = 0}$$

Here, $\cancel{\delta\sqrt{-g} = \frac{1}{2}\frac{1}{\sqrt{-g}}\delta\left[\det(g_{\mu\nu})\right]}$ ← Involves

EoM wrt $\phi(x)$?

$$\delta S = \int d^4x \sqrt{g}\, \delta\mathcal{L} = 0$$

$$\Rightarrow \int d^4x \sqrt{g} \left[\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi) \right] = 0$$

$$\Rightarrow \int d^4x \left[-\underbrace{\frac{\partial}{\partial x^\mu}\left(\sqrt{g}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)} + \sqrt{g}\frac{\partial\mathcal{L}}{\partial\varphi} \right]\delta\varphi = 0$$

A problem! / doesn't happen in flat spacetime.

→ To deal with this issue, we recall:

$$\boxed{\nabla_\mu g_{\rho\sigma} = 0} \leftarrow \text{in all frames.}$$

⇒ We upgrade all std. derivatives to covariant derivatives in going from flat to curved spacetime.

$$\partial_\mu \longrightarrow \nabla_\mu$$

$$\therefore \delta S = \int d^4x \sqrt{g} \left[-\nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\right) + \frac{\partial\mathcal{L}}{\partial\varphi} \right]\delta\varphi = 0$$

$$\Rightarrow \boxed{\nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0}$$

← Euler-Lagrange eq. in curved spacetime!! (for a matter field $\varphi(x)$)

Note:

→ Such terms ($\sim \sqrt{g}\,\mathcal{L}_m$) are minimally coupled to gravity.

when the coupling only involves $\sqrt{g}$.

→ A non-minimal coupling to gravity is when derivatives of metric are involved (coupled) with matter fields.

→ The E-L eq. we just derived is the classical field ~~the~~ equation in curved spacetime.

→ $\varphi(x)$ represents an arbitrary "field" — an analytic fn. of space-time that describes particles in QFT, & interactions or matter (extended) in CFT.

Now, let's move to the E-H action.

What is the eq. of motion (E-L eq.) wrt $g_{\mu\nu}(x)$?

→ Let's variate S_g (and S_m) wrt $g_{\mu\nu}$!

$\boxed{\text{H/W:}}$ Derive E-L eq. for $\boxed{\mathcal{L}_m = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi)}$

Variation of Einstein-Hilbert action:

$$S_g = \int \sqrt{-g}\, R\, d^4x$$

$$\Rightarrow \delta S_g = \int d^4x \left[\delta\sqrt{-g}\, R + \sqrt{-g}\, \delta\left(R_{\mu\nu} g^{\mu\nu}\right)\right] = 0$$

$$= \int d^4x \left[\underbrace{\sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu}}_{\delta S_1} + \underbrace{\sqrt{-g}\, R_{\mu\nu} \delta g^{\mu\nu}}_{\delta S_2} + \underbrace{R\,\delta\sqrt{-g}}_{\delta S_3} \right] = 0$$

$$\delta S_1 = \int d^4x \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu}$$

Now, $\delta R_{\mu\nu} = \delta R^\rho{}_{\mu\rho\nu}$

From tensor calculus: (We use established results)

$$\delta R^\rho{}_{\mu\lambda\nu} = \nabla_\lambda \left(\delta \Gamma^\rho_{\nu\mu}\right) - \nabla_\nu \left(\delta \Gamma^\rho_{\lambda\mu}\right)$$

$$\Rightarrow \delta R_{\mu\nu} = \delta R^\rho{}_{\mu\rho\nu} = \nabla_\rho \left(\delta \Gamma^\rho_{\nu\mu}\right) - \nabla_\nu \left(\delta \Gamma^\rho_{\rho\mu}\right)$$

$$\Rightarrow \delta S_1 = \int d^4x \sqrt{-g}\, \nabla_\sigma \left[g^{\mu\nu} \left(\delta \Gamma^\sigma_{\mu\nu}\right) - g^{\mu\sigma} \left(\delta \Gamma^\lambda_{\lambda\mu}\right)\right]$$

$$\delta\Gamma^{\sigma}_{\mu\nu} = -\frac{1}{2}\left[g_{\lambda\mu}\nabla_{\nu}(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\nabla_{\mu}(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^{\sigma}(\delta g^{\alpha\beta})\right]$$

— (result #2)

$$\therefore \boxed{\delta S_1 = \int d^4x \sqrt{-g}\, \nabla_{\sigma}\left[g_{\mu\nu}\nabla^{\sigma}(\delta g^{\mu\nu}) - \nabla_{\lambda}(\delta g^{\sigma\lambda})\right]} \boxed{= 0!}$$

Total derivative term!!

Now,

$$\delta S_3 = \int d^4x\, R\,\delta\sqrt{-g} = \int d^4x\, R\left[-\frac{1}{2}\frac{1}{\sqrt{-g}}\,\delta g\right]$$

Here, to make sense of δg, let's use the matrix identity:

$$\rightarrow \boxed{\begin{aligned} \ln(\det M) &= \ln\Big(\prod_j m_j\Big) = \sum_j \ln(m_j) \\ &= \mathrm{Tr}(\ln M) \end{aligned}}$$

$$\Rightarrow \quad \delta \ln(\det M) = \delta\,\mathrm{Tr}(\ln M)$$

$$\Rightarrow \quad \frac{1}{\det(M)}\cdot\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M)$$

$\sum$ all indices

$$\Rightarrow \quad \boxed{\delta(\det M) = \det(M)\,\mathrm{Tr}(\underbrace{M^{-1}}_{g^{\mu\nu}}\,\delta \underbrace{M}_{g_{\mu\nu}})}$$

$$\Rightarrow \quad \boxed{\delta g = g\left(g^{\mu\nu}\delta g_{\mu\nu}\right)} = -g\left(g_{\mu\nu}\delta g^{\mu\nu}\right)$$

$$\cancel{\Rightarrow \delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g}$$

→ Here, we've used:

$$g^{\mu\lambda}g_{\lambda\nu} = \delta^{\mu}{}_{\nu}$$

$$\Rightarrow \delta(g^{\mu\lambda}g_{\lambda\nu}) = 0$$

$$\Rightarrow \boxed{\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}}$$

$$\therefore \delta\sqrt{-g} = +\frac{1}{2}\frac{g}{\sqrt{-g}}g_{\mu\nu}\delta g^{\mu\nu}$$

$$= -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}$$

Finally, $\underline{\delta S_1 + \delta S_2 + \delta S_3}$:

$$\Rightarrow \delta S_g = \int d^4x\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]\delta g^{\mu\nu} = 0$$

$$\Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0}$$

← The Einstein eq. in absence of matter. (Vacuum)

$$\Updownarrow$$

$$\frac{1}{\sqrt{-g}}\frac{\delta S_g}{\delta g_{\mu\nu}}$$

(Q) What happens when adding S_m? $\longrightarrow \underline{16\pi G S_m}$

$$\rightarrow \boxed{8\pi G\, T_{\mu\nu}} \longrightarrow T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$$