Power Flow Studies

- Power flow studies are of great importance in planning and operation.
- A power flow study gives the magnitude and angle of the voltage at each bus.
- Once the bus voltage magnitudes and angles are known, the real and reactive power flow through each line can be computed and hence losses in a system.
- Power flow studies are a steady state analysis of a power system. They are called as **load flow studies**.
- Since the loads are specified in terms of power, the resulting equations are non-linear algebraic which need to be solved iteratively.
- We use numerical methods such as **Gauss-Seidal** and **Newton-Raphson** Methods for solving them.

Power Flow Problem:

Let V_i be the voltage at i^{th} bus.

$$V_i = |V_i|\angle\delta_i$$

Let Y_{ii} and Y_{ij} be

$$Y_{ii} = |Y_{ii}|\angle\theta_{ii} \quad Y_{ij} = |Y_{ij}|\angle\theta_{ij}$$

The net current injected into the network at bus ⓘ is

$$I_i = Y_{i1}V_1 + Y_{i2}V_2 + \cdots + Y_{iN}V_N = \sum_{n=1}^{N} Y_{in}V_n$$

where N be the total number of buses in the network. Let P_i and Q_i be the net real and reactive power entering the network at the bus ⓘ.

$$P_i + \jmath Q_i = V_i I_i^*$$

$$P_i - \jmath Q_i = V_i^* I_i$$

$$P_i - \jmath Q_i = V_i^* \sum_{n=1}^{N} Y_{in} V_n$$

On substitution,

$$P_i - \jmath Q_i = |V_i| \underline{/-\delta_i} \sum_{n=1}^{N} |Y_{in}||V_n| \underline{/\theta_{in} + \delta_n}$$

$$P_i - \jmath Q_i = \sum_{n=1}^{N} |Y_{in}||V_i||V_n| \underline{/\theta_{in} + \delta_n - \delta_i}$$

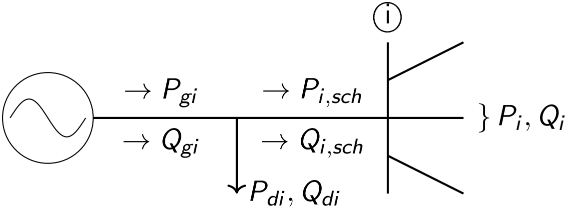
Equating real and and imaginary parts,

$$P_i = \sum_{n=1}^{N} |Y_{in}||V_i||V_n| \cos(\theta_{in} + \delta_n - \delta_i)$$

$$Q_i = -\sum_{n=1}^{N} |Y_{in}||V_i||V_n| \sin(\theta_{in} + \delta_n - \delta_i)$$

The above equations are power flow equations in the polar form. They are non linear functions of $|V|$ and δ.

$$P = f_1(|V|, \delta)$$
$$Q = f_2(|V|, \delta)$$



The net scheduled power injected into the network

$$P_{i,sch} = P_{gi} - P_{di}; \quad Q_{i,sch} = Q_{gi} - Q_{di}$$

The mismatch power

$$\Delta P_i = P_{i,sch} - P_i \quad \Delta Q_i = Q_{i,sch} - Q_i$$

If the calculated values match the scheduled values perfectly, the mismatches are zero at bus ⓘ. The power balance equations

$$(P_{gi} - P_{di}) - P_i = 0$$

$$(Q_{gi} - Q_{di}) - Q_i = 0$$

At each bus ⓘ, there are four quantities.

$$P_i,\ Q_i,\ |V_i|,\ \delta_i$$

Two of the four quantities are specified and the remaining two are calculated.

Types of Buses:

The main objective of the power flow is to find $|V|$ and δ of each bus when power generation and loads are specified. To facilitate this, the buses are classified into three different types.

1. Load buses (PQ)
2. Generator buses / Voltage controlled buses (PV)
3. Slack /Swing bus

1. Load buses:
   At each load bus, P_{gi} and Q_{gi} are zero. $P_{di}$and Q_{di} are specified.
   $$P_{i,sch} = -P_{di}; \quad Q_{i,sch} = -Q_{di}$$
   $|V_i|$ and δ_i are to be determined at each load bus.
2. Voltage-controlled buses:
   - A bus at which the voltage magnitude is kept constant is said to be voltage controlled.
   - At each generator bus, P_{gi} and $|V_i|$ are specified. Q_{gi} and δ_i are to be determined.
   - These buses are called as generator buses.
   - Buses which have voltage control capability even without generators are designated as voltage controlled buses. $P_{gi} = 0$ for those buses.

3. Slack bus:
   - For convenience, bus ① is almost always designated as the slack bus.
   - $|V_1|$ and δ_1 are specified. Usually $\delta_1 = 0^\circ$.
   - P_1 and Q_1 are to be found.
   - Since the losses can be found only after $|V|$ and δ are known at each bus, the slack bus P_g and Q_g are not specified.
   - The slack bus supplies the difference between the specified power going into the network at all the other buses and the total output plus losses.
   - A generator bus with the largest generating capacity is usually selected as the slack bus.

Power Flow Studies - Methods

- The power flow equations are nonlinear functions of $|V_i|$ and δ_i.
- Hence, iterative techniques such as Gauss-Seidel and Newton-Raphson methods are used to solve them.
- Gauss- Seidel method solves the power flow equations in rectangular form.
- Newton- Raphson method solves the power flow equations in polar form.

Gauss Seidel Method

Let $f(x)$ be a non-linear function of one variable.

$$f(x) = 0$$

To solve it, let us rewrite $f(x)$ as follows:

$$x = g(x)$$

where g is another function of x. (It need not be unique.)

- Assume x^0.
- Set $k = 0$.
- Find the next iterate

$$x^{k+1} = g(x^k)$$

- Check for convergence i.e., $|x^{k+1} - x^k| \leq \epsilon$. Otherwise repeat the above step.

Let us extend it to multi- variable functions.

$$f_1(x_1, x_2) = 0$$

$$f_2(x_1, x_2) = 0$$

- Assume x_1^0 and x_2^0.
- Set $k = 0$.
- Find the next iterate

$$x_1^{k+1} = g_1(x_1^k, x_2^k)$$

$$x_2^{k+1} = g_2(x_1^{k+1}, x_2^k)$$

- Check for convergence i.e., $\|\mathbf{x^{k+1}} - \mathbf{x^k}\| \leq \epsilon$. Otherwise repeat the above step.
- G-S method is slow. It converges linearly.
- To accelerate convergence, an acceleration factor (α) is used.

$$\mathbf{x_{acc}^{k+1}} = \mathbf{x^k} + \alpha(\mathbf{x^{k+1}} - \mathbf{x^k})$$

In power flow studies, α is 1.6.

Example : Use G-S method to solve the following equation.
Assume $x^0 = 2$. $\alpha = 1.6$

$$f(x) = x^3 - 6x^2 + 9x - 4 = 0$$

This can be written as

$$x = -\frac{1}{9}(x^3 - 6x^2 - 4)$$

First Iteration:

$$x^1 = 2.2222$$

$$x_{acc}^1 = x^0 + \alpha(x^1 - x^0) = 2 + 1.6 \times (2.2222 - 2) = 2.3556$$

It took almost 10 iterations to reach a solution.

$$x_{acc}^{10} \approx 4$$

- G-S Method is slow.
- It requires less computations per iteration.

Example : Use Gauss-Seidal method to find the solution of the following equations

$$x_1 + x_1 x_2 = 10$$
$$x_1 + x_2 = 6$$

with $x_1^0 = 1$ and $x_2^0 = 1$.
The functions can be rewritten as

$$x_1 = \frac{10}{1 + x_2}$$
$$x_2 = 6 - x_1$$

The general form of G-S Method

$$x_1^{k+1} = \frac{10}{1 + x_2^k}$$
$$x_2^{k+1} = 6 - x_1^{k+1}$$

First Iteration:

$$x_1^1 = 10/2 = 5$$
$$x_2^1 = 6 - 5 = 1$$

It took 2 iterations to reach the solution.

$$\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Suppose we start with $\mathbf{x^0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We will reach the following solution in 15 iterations.

$$\mathbf{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Gauss-Seidel Method - Power Flow Studies:

$$P_{i,sch} - \jmath Q_{i,sch} = V_i^* \sum_{n=1}^{N} Y_{in} V_n$$

where $P_{i,sch} = P_{gi} - P_{di}$ and $Q_{i,sch} = Q_{gi} - Q_{di}$.

$$\frac{P_{i,sch} - \jmath Q_{i,sch}}{V_i^*} = Y_{i1} V_1 + \cdots + Y_{ii} V_i + \cdots + Y_{iN} V_N$$

It can be rewritten as

$$V_i = \frac{1}{Y_{ii}} \left(\frac{P_{i,sch} - \jmath Q_{i,sch}}{V_i^*} - (Y_{i1} V_1 + \cdots + Y_{iN} V_N) \right)$$

$$V_i = \frac{1}{Y_{ii}} \left(\frac{P_{i,sch} - \jmath Q_{i,sch}}{V_i^*} - \sum_{\substack{n=1 \\ n \neq i}}^{N} Y_{in} V_n \right)$$

In general,

$$V_i^{k+1} = \frac{1}{Y_{ii}} \left(\frac{P_{i,sch} - \jmath Q_{i,sch}}{(V_i^k)^*} - \sum_{n=1}^{i-1} Y_{in} V_n^{k+1} - \sum_{n=i+1}^{N} Y_{in} V_n^k \right)$$

When a system has only PQ (load) buses,

1. Form $\mathbf{Y}_{bus}$ matrix from the line data.
2. Assume $V_i = 1 + \jmath 0$ for all PQ buses. It is called a *flat start*
3. Update voltages of load buses using the following equation.

$$V_i^{k+1} = \frac{1}{Y_{ii}} \left(\frac{P_{i,sch} - \jmath Q_{i,sch}}{(V_i^k)^*} - \sum_{n=1}^{i-1} Y_{in} V_n^{k+1} - \sum_{n=i+1}^{N} Y_{in} V_n^k \right)$$

4. Accelerate the voltage to improve convergence.

$$V_{i,acc}^{k+1} = V_i^k + \alpha (V_i^{k+1} - V_i^k)$$

5. Check for convergence $|V_{i,acc}^{k+1} - V_i^k| \leq \epsilon$.
6. If yes, stop. Otherwise go to step 3.

When a system has both PQ and PV (voltage controlled) buses,

1. Form $\mathbf{Y}_{bus}$ matrix from the line data.
2. Assume $V_i = 1 + \jmath 0$ for all load buses. $\delta_i = 0$ for all PV buses.
3. If the bus is PQ,
   - Update voltages of load buses using the following equation.

$$V_i^{k+1} = \frac{1}{Y_{ii}} \left(\frac{P_{i,sch} - \jmath Q_{i,sch}}{(V_i^k)^*} - \sum_{n=1}^{i-1} Y_{in} V_n^{k+1} - \sum_{n=i+1}^{N} Y_{in} V_n^k \right)$$

4. If the bus is PV, (Q_i is not given in PV buses.)
   - Find Q_i using the following equation

$$Q_i^{k+1} = -Imag \left\{ (V_i^k)^* \left(\sum_{n=1}^{i-1} Y_{in} V_n^{k+1} + \sum_{n=i}^{N} Y_{in} V_n^k \right) \right\}$$

- If $Q_{i,min} \leq Q_i^{k+1} \leq Q_{i,max}$, update the voltage as follows:

$$V_i^{k+1} = \frac{1}{Y_{ii}} \left(\frac{P_{i,sch} - \jmath Q_i^{k+1}}{(V_i^k)^*} - \sum_{n=1}^{i-1} Y_{in} V_n^{k+1} - \sum_{n=i+1}^{N} Y_{in} V_n^k \right)$$

  Since $|V_i|$ is given,

$$V_{i,corr}^{k+1} = |V_i| \frac{V_i^{k+1}}{|V_i^{k+1}|}$$

- Else, set Q_i to the violated limit.

$$Q_i = Q_{i,min} \text{ if } Q_i < Q_{i,min}$$

$$Q_i = Q_{i,max} \text{ if } Q_i > Q_{i,max}$$

  Treat the bus as PQ bus and update the voltage (both magnitude and angle) accordingly for this iteration.

5. Accelerate the voltage to improve convergence.

$$V_{i,acc}^{k+1} = V_i^k + \alpha(V_i^{k+1} - V_i^k)$$

6. Check for convergence $|V_{i,acc}^{k+1} - V_i^k| \leq \epsilon$.
7. If yes, stop. Otherwise go to step 3.
8. Find the line flows, losses and the slack bus power.

Example :

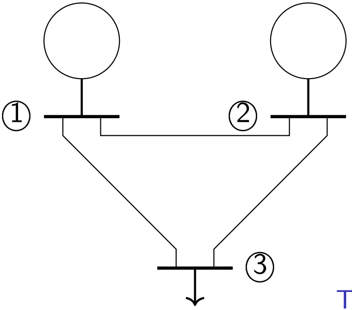


Table: Line data

Bus Code	R (p.u.)	X (p.u.)	$\frac{Y}{2}$ (p.u.)
1-2	0	$\jmath$ 0.1	$\jmath$ 0.01
1-3	0	$\jmath$ 0.1	$\jmath$ 0.01
2-3	0	$\jmath$ 0.1	$\jmath$ 0.01

Table: Bus data

Bus	P_g	Q_g	P_d	Q_d	$\|V\|$	δ	Type
1	—	—	—	—	1	0	Slack
2	0.6661	—	—	—	1.05	—	PV
3	—	—	2.8653	1.2244	—	—	PQ

Assume a flat voltage start, determine the voltage at the end of first iteration using G-S method. $0.2 \leq Q_2 \leq 2$. Take $\alpha = 1.6$.

1. Form $\mathbf{Y}_{bus}$ matrix.

$$\mathbf{Y}_{bus} = \begin{bmatrix} \frac{1}{\jmath 0.1} + \frac{1}{\jmath 0.1} + \jmath 0.01 + \jmath 0.01 & -\frac{1}{\jmath 0.1} & -\frac{1}{\jmath 0.1} \\ -\frac{1}{\jmath 0.1} & \frac{1}{\jmath 0.1} + \frac{1}{\jmath 0.1} + \jmath 0.01 + \jmath 0.01 & -\frac{1}{\jmath 0.1} \\ -\frac{1}{\jmath 0.1} & -\frac{1}{\jmath 0.1} & \frac{1}{\jmath 0.1} + \frac{1}{\jmath 0.1} + \jmath 0.01 + \jmath 0.01 \end{bmatrix}$$

$$\mathbf{Y}_{bus} = \begin{bmatrix} -\jmath 19.98 & \jmath 10 & \jmath 10 \\ \jmath 10 & -\jmath 19.98 & \jmath 10 \\ \jmath 10 & \jmath 10 & -\jmath 19.98 \end{bmatrix}$$

2. Start the first iteration $k = 0$.
   - Bus ② is a PV bus.
     - Find Q_2.

$$Q_2^1 = -Imag\left\{(V_2^0)^* \left(Y_{21}V_1^1 + Y_{22}V_2^0 + Y_{23}V_3^0\right)\right\}$$

$$= -Imag\{(1.05+\jmath 0)^* \times (\jmath 10(1+\jmath 0) + (-\jmath 19.98)(1.05+\jmath 0) + \jmath 10(1+\jmath 0))\}$$

$$Q_2^1 = 1.028$$

Q_2 is within limit. Modify V_2.

$$V_2^1 = \frac{1}{Y_{22}}\left(\frac{P_{2,sch} - \jmath Q_2^1}{(V_2^0)^*} - Y_{21}V_1^1 - Y_{23}V_3^0\right)$$

$$V_2^1 = \frac{1}{-\jmath 19.98}\left(\frac{0.6661 - \jmath 1.028}{(1.05 + \jmath 0)^*} - \jmath 10 \times (1 + \jmath 0) - \jmath 10 \times (1 + \jmath 0)\right)$$

$$V_2^1 = 1.0500 + \jmath 0.0318$$

Since $|V_2|$ is fixed,

$$V_{2,corr}^1 = |V_2|\frac{V_2^1}{|V_2^1|} = 1.05 \times \frac{1.0500 + \jmath 0.0318}{1.0505} = 1.0495 + \jmath 0.0317$$

$$V_{2,corr}^1 = 1.05\underline{/1.732^\circ}$$

- Bus ③ is a PQ bus.
  - Update V_3.

$$V_3^1 = \frac{1}{Y_{33}}\left(\frac{P_{3,sch} - \jmath Q_{3,sch}}{(V_3^0)^*} - Y_{31}V_1^1 - Y_{32}V_2^1\right)$$

$$V_3^1 = \frac{1}{-\jmath 19.98}\{\frac{-2.8653 + \jmath 1.2244}{(1+\jmath 0)^*} - \jmath 10 \times (1 + \jmath 0) \\ - \jmath 10 \times (1.0495 + \jmath 0.0317)\}$$

$$V_3^1 = 0.9645 - \jmath 0.1275$$

$$V_{3,acc}^1 = V_3^0 + \alpha(V_3^1 - V_3^0) = (1+\jmath 0) + 1.6 \times (0.9645 - \jmath 0.1275 - 1 - \jmath 0)$$

$$V_{3,acc}^1 = 0.9432 - \jmath 0.2040$$

3. Slack bus power
   It took 13 iterations to converge.

$$V = \begin{bmatrix} 1.0000 + \jmath 0.0000 \\ 1.0486 - \jmath 0.0551 \\ 0.9354 - \jmath 0.1648 \end{bmatrix}$$

$$S_1 = V_1 \times I_1^*$$

$$P_1 = 2.1985; \; Q_1 = 0.1408$$

$$P_L = 0; \; Q_L = 0.5601$$

Newton's Method :

Let $f(x)$ be a non-linear function of one variable.

$$f(x) = 0$$

- It approximates the given function as linear and solves repeatedly.

Let x^0 be the initial guess and Δx^0 be the correction value to be added to x^0 to get the actual solution.

$$f(x^0 + \Delta x^0) = 0$$

Let us expand it using Taylor's series and neglect higher order terms .

$$f(x^0) + \left.\frac{df}{dx}\right|_{x=x^0} \Delta x^0 + \cdots \approx 0$$

If we neglect higher order terms,

$$\Delta x^0 = -\frac{f(x^0)}{\left.\frac{df}{dx}\right|_{x=x^0}}$$

To get the second approximation,

$$x^1 = x^0 + \Delta x^0 = x^0 - \frac{f(x^0)}{\left.\frac{df}{dx}\right|_{x=x^0}}$$

In general

$$x^{k+1} = x^k - \frac{f(x^k)}{\left.\frac{df}{dx}\right|_{x=x^k}}$$

Check for convergence i.e., $|\mathbf{x^{k+1}} - \mathbf{x^k}| \leq \epsilon$. Otherwise repeat the above step.

Example 2: Let us solve the example 1 by Newton's Method.

$$f(x) = x^3 - 6x^2 + 9x - 4 = 0$$

$$\frac{df}{dx} = 3x^2 - 12x + 9$$

Assume $x^0 = 2$.

$$\Delta x^0 = -\frac{-2}{-3} = -0.6667$$

$$x^1 = x^0 + \Delta x^0 = 2 - 0.6667 = 1.3333$$

It took 10 iterations to reach another solution.

$$x^{10} = 1.006$$

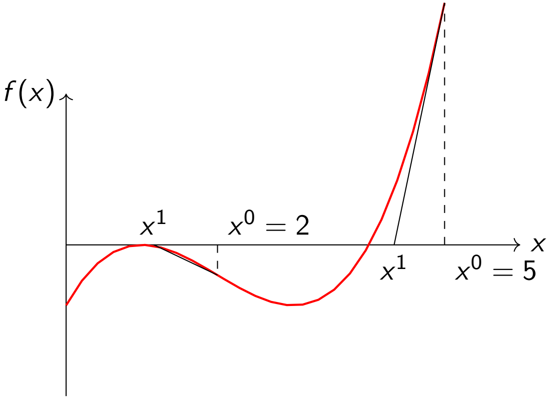
Assume $x^0 = 5$.

$$\Delta x^0 = -\frac{-2}{-3} = -0.6667$$

$$x^1 = x^0 + \Delta x^0 = 5 - 0.6667 = 4.3333$$

It took 5 iterations to reach the first solution.

$$x^5 = 4$$



If $x^0 = 3$, Newton's method will fail for this problem because $\dfrac{df}{dx} = 0$ at $x^0 = 3$.

Newton-Raphson Method:

$$f_1(x_1, x_2) = 0$$

$$f_2(x_1, x_2) = 0$$

Let $\mathbf{x^0}$ be the initial guess and $\mathbf{\Delta x^0}$ be the correction value to be added to x^0 to get the actual solution.

$$f_1(x_1^0 + \Delta x_1^0, x_2^0 + \Delta x_2^0) = 0$$

$$f_2(x_1^0 + \Delta x_1^0, x_2^0 + \Delta x_2^0) = 0$$

Let us expand it using Taylor's series and neglect higher order terms.

$$f_1(x_1^0, x_2^0) + \left.\frac{\partial f_1}{\partial x_1}\right|_{(x_1^0, x_2^0)} \Delta x_1^0 + \left.\frac{\partial f_1}{\partial x_2}\right|_{(x_1^0, x_2^0)} \Delta x_2^0 + \cdots \approx 0$$

$$f_2(x_1^0, x_2^0) + \left.\frac{\partial f_2}{\partial x_1}\right|_{(x_1^0, x_2^0)} \Delta x_1^0 + \left.\frac{\partial f_2}{\partial x_2}\right|_{(x_1^0, x_2^0)} \Delta x_2^0 + \cdots \approx 0$$

If we neglect higher order terms,

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \end{bmatrix}_{(x_1^0, x_2^0)} \begin{bmatrix} \Delta x_1^0 \\ \Delta x_2^0 \end{bmatrix} = \begin{bmatrix} 0 - f_1(x_1^0, x_2^0) \\ 0 - f_2(x_1^0, x_2^0) \end{bmatrix}$$

The *Jacobian* matrix is

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \end{bmatrix}$$

The mismatch vector $\mathbf{\Delta f}$ is

$$\mathbf{\Delta f} = \begin{bmatrix} 0 - f_1(x_1, x_2) \\ 0 - f_2(x_1, x_2) \end{bmatrix}$$

Therefore, the mismatch equations are

$$\mathbf{J}^0 \mathbf{\Delta x^0} = \mathbf{\Delta f^0}$$

By solving for $\mathbf{\Delta x^0}$, we get the next estimates.

$$\mathbf{x^1} = \mathbf{x^0} + \mathbf{\Delta x^0}$$

In general

$$\mathbf{x^{k+1}} = \mathbf{x^k} + \mathbf{\Delta x^k}$$

The above process is repeated till $\|\mathbf{\Delta x^k}\| \leq \epsilon$.

- Every iteration, $\mathbf{J}$ has to be formed.
- $\mathbf{\Delta x}$ has to be found by solving the mismatch equations either by inverse or triangular factorization.

Example 3: Consider the following nonlinear equations.

$$\begin{aligned} 4x_2 \sin x_1 &= -0.6 \\ 4x_2^2 - 4x_2 \cos x_1 &= -0.3 \end{aligned}$$

Find x_1 and x_2. Assume $x_1^0 = 0$ rad and $x_2^0 = 1$.

$$\Delta \mathbf{f} = \begin{bmatrix} -0.6 - 4x_2 \sin x_1 \\ -0.3 - 4x_2^2 + 4x_2 \cos x_1 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_2 \cos x_1 & 4 \sin x_1 \\ 4x_2 \sin x_1 & 8x_2 - 4 \cos x_1 \end{bmatrix}$$

First Iteration:

$$\mathbf{J}^0 \mathbf{\Delta x^0} = \mathbf{\Delta f^0}$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \Delta x_1^0 \\ \Delta x_2^0 \end{bmatrix} = \begin{bmatrix} -0.6 \\ -0.3 \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_1^0 \\ \Delta x_2^0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -0.6 \\ -0.3 \end{bmatrix} = \begin{bmatrix} -0.150 \\ -0.075 \end{bmatrix}$$

$$\begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix} + \begin{bmatrix} \Delta x_1^0 \\ \Delta x_2^0 \end{bmatrix} = \begin{bmatrix} -0.15 \\ 0.925 \end{bmatrix}$$

It converged after 3 iterations.

$$x_1^4 = -0.1668 \text{ rad}; \quad x_2^4 = 0.9030$$

- N-R method converges fast if the starting point is near a solution.
- It takes a few iterations to converge irrespective of system variables.
- But it requires a lot of computations per iteration (Calculation of $\mathbf{J}$ and either inverse or triangular factorization).
- It does not converge to a solution from an arbitrary starting point.

N-R Method - Power Flow Studies

$$P_i = \sum_{n=1}^{N} |Y_{in}||V_i||V_n| \cos(\theta_{in} + \delta_n - \delta_i)$$
$$Q_i = -\sum_{n=1}^{N} |Y_{in}||V_i||V_n| \sin(\theta_{in} + \delta_n - \delta_i)$$

They are non linear functions of $|V|$ and δ.

$$P = f_1(|V|, \delta)$$
$$Q = f_2(|V|, \delta)$$

By N-R method, the mismatch equations are

$$\begin{bmatrix} \mathbf{\Delta P} \\ \mathbf{\Delta Q} \end{bmatrix} = \begin{bmatrix} \mathbf{J_1} & \mathbf{J_2} \\ \mathbf{J_3} & \mathbf{J_4} \end{bmatrix} \begin{bmatrix} \mathbf{\Delta}\delta \\ \mathbf{\Delta}|\mathbf{V}| \end{bmatrix}$$

where

$$\mathbf{\Delta P} = \mathbf{P}^{\text{sch}} - \mathbf{P}^{\text{cal}}; \quad \mathbf{\Delta Q} = \mathbf{Q}^{\text{sch}} - \mathbf{Q}^{\text{cal}}$$

$\mathbf{J_1}$, $\mathbf{J_2}$, $\mathbf{J_3}$ and $\mathbf{J_4}$ are sub matrices of Jacobian.

$$\mathbf{J_1} = [\frac{\partial P}{\partial \delta}]; \ \mathbf{J_2} = [\frac{\partial P}{\partial |V|}]; \ \mathbf{J_3} = [\frac{\partial Q}{\partial \delta}]; \ \mathbf{J_4} = [\frac{\partial Q}{\partial |V|}];$$

To find the size of the Jacobian matrix, let us assume that there are m voltage controlled buses in the system of n buses.

- Since P is specified for $n-1$ buses, the size of $\mathbf{\Delta P}$ is $(n-1) \times 1$.
- Since Q is specified for only PQ buses, the size of $\mathbf{\Delta Q}$ is $(n-m-1) \times 1$.
- Size of $\mathbf{J_1}$ is $(n-1) \times (n-1)$
- Size of $\mathbf{J_2}$ is $(n-1) \times (n-m-1)$
- Size of $\mathbf{J_3}$ is $(n-m-1) \times (n-1)$
- Size of $\mathbf{J_4}$ is $(n-m-1) \times (n-m-1)$

The size of the Jacobian is $(2n-m-2) \times (2n-m-2)$.

To find $\mathbf{J_1}$:

Off-Diagonal Elements:

$$\frac{\partial P_i}{\partial \delta_n} = -|Y_{in}||V_i||V_n| \sin(\theta_{in} + \delta_n - \delta_i) \quad i \neq n$$

Diagonal Elements :

$$\frac{\partial P_i}{\partial \delta_i} = \sum_{\substack{n=1 \\ n \neq i}}^{N} |Y_{in}||V_i||V_n| \sin(\theta_{in} + \delta_n - \delta_i)$$

To find $\mathbf{J_2}$:

Off-Diagonal Elements:

$$\frac{\partial P_i}{\partial |V_n|} = |Y_{in}||V_i| \cos(\theta_{in} + \delta_n - \delta_i) \quad i \neq n$$

Diagonal Elements:

$$\frac{\partial P_i}{\partial |V_i|} = 2|Y_{ii}||V_i| \cos(\theta_{ii}) + \sum_{\substack{n=1 \\ n \neq i}}^{N} |Y_{in}||V_n| \cos(\theta_{in} + \delta_n - \delta_i)$$

To find $\mathbf{J_3}$:
Off-Diagonal Elements:

$$\frac{\partial Q_i}{\partial \delta_n} = -|Y_{in}||V_i||V_n| \cos(\theta_{in} + \delta_n - \delta_i) \quad i \neq n$$

Diagonal Elements:

$$\frac{\partial Q_i}{\partial \delta_i} = \sum_{\substack{n=1 \\ n \neq i}}^{N} |Y_{in}||V_i||V_n| \cos(\theta_{in} + \delta_n - \delta_i)$$

To find $\mathbf{J_4}$:
Off-Diagonal Elements:

$$\frac{\partial Q_i}{\partial |V_n|} = -|Y_{in}||V_i| \sin(\theta_{in} + \delta_n - \delta_i) \quad i \neq n$$

Diagonal Elements:

$$\frac{\partial Q_i}{\partial |V_i|} = -2|V_i||Y_{ii}| \sin(\theta_{ii}) - \sum_{\substack{n=1 \\ n \neq i}}^{N} |Y_{in}||V_n| \sin(\theta_{in} + \delta_n - \delta_i)$$

Example :

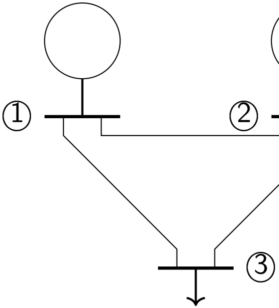


Table: Line data

Bus Code	R (p.u.)	X (p.u.)	$\frac{Y}{2}$ (p.u.)
1-2	0	$\jmath\,0.1$	$\jmath\,0.01$
1-3	0	$\jmath\,0.1$	$\jmath\,0.01$
2-3	0	$\jmath\,0.1$	$\jmath\,0.01$

Table: Bus data

Bus	P_g	Q_g	P_d	Q_d	$\|V\|$	δ	Type
1	—	—	—	—	1	0	Slack
2	0.6661	—	—	—	1.05	—	PV
3	—	—	2.8653	1.2244	—	—	PQ

Assume a flat voltage start, determine the voltage at the end of first iteration using N-R method. $0.2 \leq Q_2 \leq 2$.

② is a PV bus and ③ is a PQ bus.

$$P_2 = |Y_{21}||V_2||V_1|\cos(\theta_{21} + \delta_1 - \delta_2) + |Y_{22}||V_2|^2 \cos(\theta_{22}) + |Y_{23}||V_2||V_3|\cos(\theta_{23} + \delta_3 - \delta_2)$$

$$P_3 = |Y_{31}||V_3||V_1|\cos(\theta_{31}+\delta_1-\delta_3)+|Y_{32}||V_3||V_2|\cos(\theta_{32}+\delta_2-\delta_3) + |Y_{33}||V_3|^2 \cos(\theta_{33})$$

$$Q_3 = -|Y_{31}||V_3||V_1|\sin(\theta_{31}+\delta_1-\delta_3)-|Y_{32}||V_3||V_2|\sin(\theta_{32}+\delta_2-\delta_3) - |Y_{33}||V_3|^2 \sin(\theta_{33})$$

Since, $n = 3$ and $m = 1$, the mismatch equations are

$$\begin{bmatrix} \Delta P_2 \\ \Delta P_3 \\ \Delta Q_3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} \frac{\partial P_2}{\partial \delta_2} & \frac{\partial P_2}{\partial \delta_3} & \frac{\partial P_2}{\partial |V_3|} \\ \frac{\partial P_3}{\partial \delta_2} & \frac{\partial P_3}{\partial \delta_3} & \frac{\partial P_3}{\partial |V_3|} \\ \frac{\partial Q_3}{\partial \delta_2} & \frac{\partial Q_3}{\partial \delta_3} & \frac{\partial Q_3}{\partial |V_3|} \end{bmatrix}_{3\times 3} \begin{bmatrix} \Delta \delta_2 \\ \Delta \delta_3 \\ \Delta |V_3| \end{bmatrix}_{3\times 1}$$

1. Form $\mathbf{Y}_{bus}$ matrix.

$$\mathbf{Y}_{bus} = \begin{bmatrix} 19.98\underline{/-90^\circ} & 10\underline{/90^\circ} & 10\underline{/90^\circ} \\ 10\underline{/90^\circ} & 19.98\underline{/-90^\circ} & 10\underline{/90^\circ} \\ 10\underline{/90^\circ} & 10\underline{/90^\circ} & 19.98\underline{/-90^\circ} \end{bmatrix}$$

2. Assume $\delta_2 = \delta_3 = 0.$ and $|V_3| = 1.$
3. Set $k = 0.$

4. Find the mismatch vector.

$$\begin{bmatrix} \Delta P_2 \\ \Delta P_3 \\ \Delta Q_3 \end{bmatrix}^0 = \begin{bmatrix} P_{2,sch} - P_2(\delta_2^0, \delta_3^0, |V_3|^0) \\ P_{3,sch} - P_3(\delta_2^0, \delta_3^0, |V_3|^0) \\ Q_{3,sch} - Q_3(\delta_2^0, \delta_3^0, |V_3|^0) \end{bmatrix} = \begin{bmatrix} 0.6661 - 0 \\ -2.8653 - 0 \\ -1.2244 - (-0.52) \end{bmatrix}$$

$$\begin{bmatrix} \Delta P_2 \\ \Delta P_3 \\ \Delta Q_3 \end{bmatrix}^0 = \begin{bmatrix} 0.6661 \\ -2.8653 \\ -0.7044 \end{bmatrix}$$

$$\mathbf{J}^0 = \begin{bmatrix} 21 & -10.5 & 0 \\ -10.5 & 20.5 & 0 \\ 0 & 0 & 19.46 \end{bmatrix}$$

5. Solve for the correction vector.

$$\begin{bmatrix} \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta|V_3| \end{bmatrix}^0 = [\mathbf{J^0}]^{-1} \begin{bmatrix} \Delta P_2 \\ \Delta P_3 \\ \Delta Q_3 \end{bmatrix}^0$$

$$\begin{bmatrix} \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta|V_3| \end{bmatrix}^0 = \begin{bmatrix} -0.0513 \text{ rad} \\ -0.1660 \text{ rad} \\ -0.0362 \end{bmatrix}$$

6. Find the new estimate.

$$\begin{bmatrix} \delta_2 \\ \delta_3 \\ |V_3| \end{bmatrix}^1 = \begin{bmatrix} \delta_2 \\ \delta_3 \\ |V_3| \end{bmatrix}^0 + \begin{bmatrix} \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta|V_3| \end{bmatrix}^0$$

$$\begin{bmatrix} \delta_2 \\ \delta_3 \\ |V_3| \end{bmatrix}^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.0513 \\ -0.1660 \\ -0.0362 \end{bmatrix} = \begin{bmatrix} -0.0513 \text{ rad} \\ -0.1660 \text{ rad} \\ 0.9638 \end{bmatrix}$$

7. k=k+1 and go back to step 4.
8. It took 5 iterations to converge.

$$\begin{bmatrix} \delta_2 \\ \delta_3 \\ |V_3| \end{bmatrix} = \begin{bmatrix} -0.0524 \text{ rad} \\ -0.1745 \text{ rad} \\ 0.9500 \end{bmatrix}$$

9. Slack bus power:

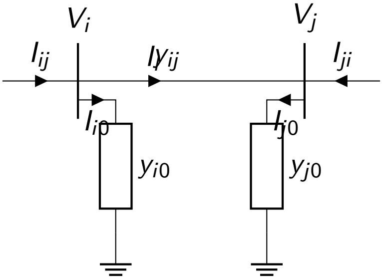
$$P_g = 2.1992; \; Q_g = 0.1387$$

10. Losses :

$$P_L = 0; \; Q_L = 0.5560$$

Line Flows, Losses and Slack Bus Power:

Once the bus voltages are found numerically, the next step is to find line flows and losses.



$$I_{ij} = I_l + I_{i0} = y_{ij}(V_i - V_j) + y_{i0}V_i$$

$$I_{ji} = -I_l + I_{j0} = y_{ij}(V_j - V_i) + y_{j0}V_j$$

The complex power S_{ij} and S_{ji} are

$$S_{ij} = V_i I_{ij}^*$$

$$S_{ji} = V_j I_{ji}^*$$

The power loss in the line $i - j$ is

$$S_{L,ij} = S_{ij} + S_{ji}$$

The slack bus power can be computed as follows:

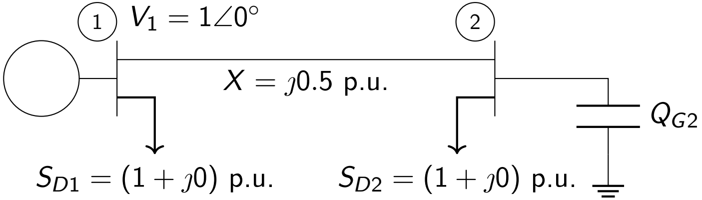
$$P_1 - \jmath Q_1 = V_1^* \sum_{n=1}^{N} Y_{1n} V_n$$

The Slack bus does the following.

$$P_{G1} = -(\sum_{\substack{i \forall Gen \\ i \neq Slack}} P_{Gi} - \sum_{i \forall load} P_{di} - P_L)$$

$$Q_{G1} = -(\sum_{\substack{i \forall Gen \\ i \neq Slack}} Q_{Gi} - \sum_{i \forall load} Q_{di} - Q_L)$$

Example : For the system shown in the figure, S_{D1} and S_{D2} are complex power demands at bus 1 and bus 2 respectively. If $|V_2| = 1$ p.u., compute the rating of capacitor (Q_{G2}) connected at bus 2 in p.u.



1. Form $\mathbf{Y}_{bus}$ matrix.

$$\mathbf{Y}_{bus} = \begin{bmatrix} \frac{1}{\jmath 0.5} & -\frac{1}{\jmath 0.5} \\ -\frac{1}{\jmath 0.5} & \frac{1}{\jmath 0.5} \end{bmatrix} = \begin{bmatrix} 2\underline{/-90^\circ} & 2\underline{/90^\circ} \\ 2\underline{/90^\circ} & 2\underline{/-90^\circ} \end{bmatrix}$$

2. Since ② is a voltage controlled (PV) bus,

$$P_{2,sch} = P_{G2} - P_{D2} = 0 - 1 = -1$$

The net real power injected into the network at ②.

$$P_2 = Y_{21}||V_2||V_1|\cos(\theta_{21} + \delta_1 - \delta_2) + Y_{22}||V_2|^2 \cos(\theta_{22})$$

δ_2 can be found directly instead of N-R method for this equation. At ②,

$$P_{2,sch} = P_2$$

On substitution,

$$2\cos(90^\circ - \delta_2) = -1$$

$$2\sin\delta_2 = -1$$

$$\delta_2 = -30^\circ$$

3. Q_{G2} is found as follows:

$$Q_{2,sch} = Q_{G2} - Q_{D2} = Q_{G2}$$

The net reactive power injected into the network at ②.

$$Q_2 = -Y_{21}||V_2||V_1|\sin(\theta_{21} + \delta_1 - \delta_2) - Y_{22}||V_2|^2 \sin(\theta_{22})$$

On substitution,

$$Q_2 = -2\sin(120^\circ) + 2 = 0.2679$$

At ②,

$$Q_{2,sch} = Q_2$$

Therefore,

$$Q_{G2} = 0.2679 \text{ p.u.}$$